

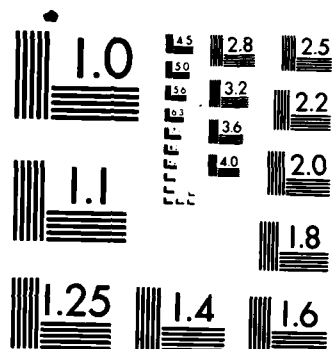
UNCLASSIFIED

DECISION AND CONTROL LAB P H MCDOWELL MAR 85 DC-77

1/2

F/G 12/1

NL



MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

12

REPORT DC-77

MARCH, 1985

**CSL COORDINATED SCIENCE LABORATORY**  
**DECISION AND CONTROL LABORATORY**

AD-A161 428

**CONTROLLER DESIGN FOR  
LINEAR STOCHASTIC SYSTEMS  
WITH UNCERTAIN PARAMETERS**



DTIC FILE COPY

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

11 18-85 021

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

DD-A161428

## REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS NONE			
2a. SECURITY CLASSIFICATION AUTHORITY N/A			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release, distribution unlimited.			
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A						
4. PERFORMING ORGANIZATION REPORT NUMBER(S) R-1038 UILU-ENG-85-2213 (DC-77)			5. MONITORING ORGANIZATION REPORT NUMBER(S) N/A			
6a. NAME OF PERFORMING ORGANIZATION Coordinated Science Laboratory, Univ. of Illinois		6b. OFFICE SYMBOL (If applicable) N/A	7a. NAME OF MONITORING ORGANIZATION Joint Services Electronics Program			
6c. ADDRESS (City, State and ZIP Code) 1101 W. Springfield Avenue Urbana, Illinois 61801			7b. ADDRESS (City, State and ZIP Code) Research Triangle Park, NC 27709			
8a. NAME OF FUNDING/SPONSORING ORGANIZATION		8b. OFFICE SYMBOL (If applicable) N/A	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER			
8c. ADDRESS (City, State and ZIP Code)			10. SOURCE OF FUNDING NOS.			
			PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
11. TITLE (Include Security Classification) Controller Design for Linear Stochastic Systems with Uncertain....			N/A	N/A	N/A	N/A
12. PERSONAL AUTHOR(S) Paul Howard McDowell						
13a. TYPE OF REPORT Technical		13b. TIME COVERED FROM _____ TO _____		14. DATE OF REPORT (Yr., Mo., Day) March 1985		15. PAGE COUNT 168
16. SUPPLEMENTARY NOTATION N/A						
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)			
FIELD	GROUP	SUB. GR.				
19. ABSTRACT (Continue on reverse if necessary and identify by block number)						
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>				21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL				22b. TELEPHONE NUMBER (Include Area Code)		22c. OFFICE SYMBOL NONE

CONTROLLER DESIGN FOR LINEAR STOCHASTIC SYSTEMS  
WITH UNCERTAIN PARAMETERS

BY

PAUL HOWARD McDOWELL

B.S., University of Illinois, 1982

THESIS

Submitted in partial fulfillment of the requirements  
for the degree of Master of Science in Electrical Engineering  
in the Graduate College of the  
University of Illinois at Urbana-Champaign, 1985

Urbana, Illinois

Accession For	
NTIS GRA&I	<input checked="checked" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

# ACKNOWLEDGEMENT

Few research projects involve the efforts of only one person. Therefore, the author would like to thank Professor Tamer Başar, first for his initial inspiration for the project, then for his constant guidance during the course of the research. Also, the Decision and Control Group should be acknowledged, since without their cumulative efforts, this author would not have the basic knowledge necessary for this type of work.

## TABLE OF CONTENTS

CHAPTER	Page
1. INTRODUCTION.....	1
2. PROBLEM STATEMENT.....	6
3. A LEAST SENSITIVE SOLUTION.....	9
3.1. Assumptions on Distributions.....	9
3.2. Restrictions on the Control Set.....	10
3.3. Problem Redefinition and Solution.....	12
3.4. Examples.....	16
3.5. Conclusions.....	39
4. A MINIMAX SOLUTION.....	44
4.1. The Maximin Solution.....	45
4.2. The Minimax Solution - Existence of a Saddle Point.....	47
4.3. Scalar Systems.....	54
4.4. Examples.....	90
4.5. Conclusions.....	105
5. A TWO-DIMENSIONAL SYSTEM.....	107
5.1. Formulation of Examples.....	108
5.2. Least Sensitive Controllers.....	109
5.3. Minimax Controllers.....	134
5.4. Conclusions.....	150
6. CONCLUSIONS.....	152
APPENDIX: A CONJUGATE GRADIENT ALGORITHM.....	157
REFERENCES.....	165

## CHAPTER 1

## INTRODUCTION

The design of optimal controllers for linear stochastic systems requires an accurate description of the system. However, the construction of an accurate model of real systems is often not possible. These inaccuracies can stem from the fact that the adopted linear model may be only a first-order approximation of a nonlinear system. Also, there may be actual uncertainty in the parameters of the real system. This type of uncertainty can arise, for instance, if one wishes to design a single type of controller for a large number of similar systems, when, for example, the system is mass produced and it is impractical to tune each controller to each system. Another situation where this type of uncertainty can arise is when the parameters of a single system vary slowly over long periods of time, perhaps due to wear or changes in the environment. These uncertainties, in this thesis, are grouped into a vector of parameters. We consider

several ways of handling these uncertainties in the design process, *as outlined below.*

*Often, it is possible to assign a prior distribution to the parameter vector. Then by considering the unknown parameter vector as a new set of system states, a nonlinear stochastic control problem can be formed. However, this type of problem may not be desirable, especially if the uncertainties were created by linearization of a nonlinear system in the first place. In this case, one may wish to build a controller that exhibits adequate performance for all values of the parameter, perhaps based on the weight each value of the parameter receives from the distribution.*



Another method is to assume that the parameters vary within a certain range about a nominal value. The use of an optimal controller for the nominal value may result in instabilities or poor performance for off-nominal values of the parameter. To design controllers for these systems, some performance must be sacrificed to desensitize the controller.

A third method of handling these uncertainties is to assume that the parameters are unknown within a given set. This set may be compact, or have a finite number of values. In many circumstances, one may design an adaptive controller that identifies the unknown parameter and tunes itself to the identified model. However, in some circumstances this identification scheme is not practical, if, for instance the parameters are slowly varying, or if the parameters jump to different values at unknown instances in time. In such cases, the identification process may not have time to converge before a new identification needs to be made. Then, it may be desirable to design a controller that does not identify the parameter, but has acceptable performance for all the values within the given set.

Any one of these methods of handling uncertainties in parameters, or a combination of them, can be considered when designing controllers which exhibit desirable insensitivity properties. This thesis will discuss two methods of designing controllers for systems whose corresponding models have uncertain parameters. We will consider linear stochastic models of the form:

$$\dot{x}_t = F(\theta)x_t + G(\theta)u_t + K\tilde{w}_t \quad (1a)$$

$$y_t = H(\theta)x_t + L\tilde{v}_t \quad (1b)$$

where  $\theta$  is a constant vector of unknown parameters, and  $\tilde{w}_t$  and  $\tilde{v}_t$  are white noise processes. A time average of a quadratic cost is to be minimized:

$$J = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x_t' W x_t + u_t' W u_t) dt \quad (2)$$

Equations (1) and (2) are not well-defined mathematically since (1) is generated by white noise, which is not a real physical process, and (2) is an integral of a stochastic process. In Chapter 2, Equation (1) is put into Ito differential form, and Equation (2) is correspondingly redefined. The exact assumptions on the unknown parameters are stated. Finally, an exact description of the problem that is solved in the design of each type of controller is stated.

The first type of controller that is considered we call a least sensitive controller since this controller minimizes the average cost over the entire parameter set based on the assumption that all parameters are equally likely. The second type of controller that is considered we call a minimax controller since a controller is sought that minimizes the worst-case cost.

In Chapter 3, we consider the unknown parameter to be in a compact set centered about a nominal value. We also assume that each value in the set is equally likely, so that a uniform distribution is induced on the set. Therefore, we use a combination of all of the three types of assumptions on the parameter set discussed above. The objective in this chapter is to design a controller that minimizes the average cost over the entire set. The optimal controller for the nominal value is a linear feedback of estimates of the full state. To facilitate the solution of this problem, we restrict

the class of controllers to the class of linear feedback controllers. However, the dimension of the controllers is not restricted to that of the dimension of the states since full-order controllers are not always required to achieve acceptable performance. Ashkenzai and Bryson [1] have presented a method for solution of this problem when discrete distributions are assumed. It is shown that this method may be extended to continuous distributions. We consider the performance of this type of controller for several examples, and these examples show that some performance must indeed be sacrificed to achieve lower parameter sensitivity.

In Chapter 4, we again assume that the unknown parameter is in a compact set centered around the nominal value. In this problem we seek to minimize the worst-case performance. A way to design these controllers is to use the optimal linear regulator for the model that exhibits the worst performance. However, it is not immediately obvious that these two problems have the same solution since the model that is least favorable for control may not be least favorable for state estimation. Looze, Poor and others [7] have shown that these two solutions are equivalent for the case where there is uncertainty in the second-order statistics of the noise processes. An attempt is made to use a similar procedure for the case now under consideration, that is, when there is uncertainty in the system dynamics. Chapter 4 explains how an error model was formed in the case of uncertainty in the noise statistics, and how this model was used to show the equivalence of the two solutions in this case. However, for the problem now under consideration, this error model cannot be formed, and thus the equivalence of the two solutions is not clear. Indeed, for the scalar case, we find parameter sets

such that the two solutions are not equivalent. Therefore, in Chapter 4, we find all the parameter sets satisfying certain convexity conditions such that the two solutions are equivalent.

The examples of Chapter 3 are then considered in Chapter 4. The least sensitive controllers of Chapter 3 are seen to exhibit superior performance over a wider range of parameter values in the sets under consideration, but, if the range of parameter values off the nominal value is large enough, the minimax controllers of Chapter 4 have a lower maximum cost than the least sensitive controllers of Chapter 3.

In Chapter 5, a second-order single-input single-output example is considered. Then, some aspects of the design and performance of the two types of controllers are discussed that are not brought out in the previous chapters. The order of the system allows us to investigate the relative performance of reduced-order least sensitive controllers with respect to the full-order least sensitive controllers and the full-order minimax controllers. This chapter shows how maximin controllers for a particular example may be designed on a numerical basis. Also, the equivalence of this solution to the minimax solution may be analyzed numerically.

Finally, in Chapter 6, the general properties of the two types of controllers are discussed. We also discuss the advantages and disadvantages of each design, and thus outline what factors are considered in choosing between the two designs. Finally, we discuss what other designs may be considered to achieve a desirable performance of the control system.

## CHAPTER 2

### PROBLEM STATEMENT

The system equations, (1), can be made precise by modelling  $\tilde{y}_t$ :  $\tilde{y}_t = \frac{dy_t}{dt}$ . Then the system equations can be put in Ito differential form:

$$dx_t = F(\theta)x_t dt + G(\theta)u_t dt + Kdw_t, \quad (3a)$$

$$dy_t = H(\theta)x_t dt + Ldv_t. \quad (3b)$$

$\begin{pmatrix} w_t \\ v_t \end{pmatrix}$  is a vector Wiener process, with

$$E \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} = \underline{0}, \quad (4a)$$

$$E \left[ \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} \begin{pmatrix} dw_t \\ dv_t \end{pmatrix}^T \right] = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} dt, \quad (4b)$$

and  $Q$  and  $R$  are positive definite.

The parameter,  $\theta$ , is constant but not known precisely. However,  $\theta$  is assumed to be contained in a nonempty, closed, compact set,  $\Theta$ . Also, the pairs,  $[F(\theta), G(\theta)]$  and  $[F(\theta), H(\theta)]$ , are assumed to be stabilizable and detectable, respectively, for each  $\theta \in \Theta$ . The problem is to choose  $u_t$  to minimize a time average of the quadratic cost functional, (2). Equation (2) becomes well-defined:

$$J(u_t, \theta) = E_{w,v} \left[ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x_t^T W_x x_t + u_t^T W_u u_t) dt \right], \quad (5)$$

where  $W_x$  is positive semidefinite and  $W_u$  is positive definite.

At time  $t$ , the controller has access to past output measurements, so that

$$u_t = \mu(t, y_s ; s \leq t) \quad , \quad t \geq 0 \quad , \quad (6)$$

where  $\mu$  is an admissible control strategy, chosen from a certain class,  $\Pi$ . Every element of  $\Pi$  must satisfy the following conditions [3]:

- a)  $\mu$  must be causal,
- b)  $\mu$  must be such that Equation (3a) has a unique solution that is sample-path continuous,
- c)  $\mu$  must be such that the cost function, (5), is well-defined.

When the parameter vector is known, the solution to this problem is well-known. The controller is an optimal linear feedback of the least-squares estimate of the plant states. Although an optimal controller can be calculated for every  $\theta \in \Theta$ , the correct controller cannot be applied without prior knowledge of  $\theta$ . Therefore, when  $\theta$  is unknown, this strategy is no longer feasible.

In the next two sections, two suboptimal solutions that are less sensitive to variations in the parameter vector are studied.

The first solution is based on the assumption that any  $\theta \in \Theta$  has an equal chance of occurring. Therefore, a uniform distribution is assumed over the set,  $\Theta$ . Then  $u_t$  is chosen to minimize the average cost over  $\Theta$ :

$$J(u_t) = E_{\theta} \{ E_{w,v} [ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x_t^T W x_t + u_t^T W u_t) dt ] \} \quad . \quad (7)$$

As a further simplification,  $u_t$  is assumed to be linear with output feedback. The problem is then reduced to a constrained optimization problem.

The second solution minimizes the worst case performance; that is,

$$J = \min_{u_t \in \Pi} \max_{\theta \in \Theta} J(u_t, \theta) . \quad (8)$$

When a saddle point exists, then (8) has the same solution as the problem,

$$J = \max_{\theta \in \Theta} \min_{u_t \in \Pi} J(u_t, \theta) . \quad (9)$$

Solution (9) is equivalent to using the optimal linear regulator for the  $\theta \in \Theta$  that exhibits the worst cost.

## CHAPTER 3

## A LEAST SENSITIVE SOLUTION

In this chapter a suboptimal solution is sought whereby a prior distribution is assumed for  $\theta$  over the set,  $\Theta$ . The problem is solved here for any continuous distribution. However, since the assumption that  $\theta$  is equally probable over the entire set is made, the uniform distribution is assumed in the examples. The problem then becomes one of choosing a control to minimize the average cost over the entire parameter set. To accomplish this purpose, the procedure described by Ashkenzai and Bryson [1] is applied to a series of discrete distributions that converge in distribution to the continuous distribution. This procedure restricts the control set to linear controllers with output feedback. The assumption of a discrete distribution, together with a linear control structure, reduces the problem to an optimization problem with equality constraints.

Two examples are then considered for scalar plant and measurement equations. One example has an unknown parameter in the plant dynamics, and the other example has an unknown parameter in the control gain. The performance of the two resulting solutions will be discussed. In particular, it is seen that disturbance attenuation must be sacrificed as more parameter insensitivity is desired.

### 3.1. Assumptions on Distributions

As discussed before, the assumption of a uniform distribution is desired since the parameter is unknown within a given set. However, here we



only assume that a continuous distribution is given. To facilitate the solution process, a series of discrete distributions that converges to the continuous distribution is created. Specifically,

$f_c(\theta)$ , a probability density function defined for  $\theta \in \Theta$

$$f_N(\theta) = \begin{cases} f_i & \theta = \theta_i \quad i=1, \dots, N \\ 0 & \text{otherwise} \end{cases}$$

and also,

$$F_N(\theta) \rightarrow F_c(\theta) \text{ as } N \rightarrow \infty,$$

where  $F_N$  and  $F_c$  are the corresponding probability distribution functions for  $f_N$  and  $f_c$ .

### 3.2. Restrictions on the Control Set

The control set is restricted to linear output feedback controllers of the form:

$$u_t = C(p)z_t \tag{10a}$$

$$\dot{z}_t = A(p)z_t + B(p)y_t \tag{10b}$$

where  $p$  is a vector of the entries in  $A(p)$ ,  $B(p)$ ,  $C(p)$ .

As discussed in Ashkenzai and Bryson [1], if all the entries of  $A$ ,  $B$ ,  $C$  were specified as parameters, the parameter vector corresponding to the minimum cost would not be unique, since only the transfer functions from  $y_t$  to  $u_t$  are of importance. Therefore, the dimension of  $p$  must be the smallest possible for the specified order of the controller and the dimensions of  $u_t$  and  $y_t$ .

To uniquely specify all the controller transfer functions, the following dimension of  $p$  is required [1]:

$$\text{DIM}(p) = [\text{DIM}(u_t) + \text{DIM}(y_t)]\text{DIM}(z_t) \quad . \quad (11)$$

In order to standardize the minimal realization of the control parameters in  $A$ ,  $B$ ,  $C$ , the following block diagonal form of the control matrices is used [1]:

$$A = \left( \begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline 0 & 0 & \ddots \end{array} \right) \quad , \quad (12a)$$

and the  $A_i$ 's are  $2 \times 2$  blocks of the form:

$$A_i = \begin{pmatrix} 0 & 1 \\ p_{A1i} & p_{A2i} \end{pmatrix} \quad (12b)$$

except when  $\text{DIM}(z_t)$  is odd, in which case the last  $A_i$  is a single scalar parameter.

$$B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \end{pmatrix} \quad , \quad (13a)$$

where

$$B_i = (p_{B1i} \quad p_{B2i} \quad \dots) \quad . \quad (13b)$$

$$C = (C_1 \mid C_2 \mid \dots) \quad (14a)$$

where each  $C_i$  has two columns:

$$C_i = \begin{pmatrix} 0 & 1 \\ p_{C1i} & p_{C2i} \\ p_{C3i} & p_{C4i} \\ \vdots & \vdots \end{pmatrix}, \quad (14b)$$

except when  $\text{DIM}(z_t)$  is odd, in which case the last  $C_i$  has only one column:

$$C_i = \begin{pmatrix} 1 \\ p_{C1i} \\ p_{C2i} \\ \vdots \end{pmatrix}. \quad (14c)$$

With the assumptions of a discrete distribution and of a linear block diagonal form of the controller, the problem can be reformulated as an optimization problem with equality constraints.

### 3.3. Problem Redefinition and Solution

The system equations, (3), together with the specified block diagonal form of the controller, (10), form a new closed loop system:

$$\begin{pmatrix} dx_t \\ dz_t \end{pmatrix} = S(\theta, p) \begin{pmatrix} x_t \\ z_t \end{pmatrix} dt + N(p) \begin{pmatrix} dw_t \\ dv_t \end{pmatrix}. \quad (15)$$

with

$$E \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} = \underline{0}, \quad (16a)$$

$$E \left[ \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} \begin{pmatrix} dw_t \\ dv_t \end{pmatrix}' \right] = V dt, \quad (16b)$$

where  $S(\theta, p)$ ,  $N(p)$  and  $V$  are given by

$$S(\theta, p) = \begin{pmatrix} F(\theta) & G(\theta)C(p) \\ B(p)H(\theta) & A(p) \end{pmatrix}, \quad (17)$$

$$N(p) = \begin{pmatrix} K & 0 \\ 0 & B(p)L \end{pmatrix}, \quad (18)$$

$$V = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}. \quad (19)$$

For discrete distributions, the cost functional becomes

$$J(p) = \sum_{i=1}^N f_i J(\theta_i, p). \quad (20)$$

Using (5),

$$J(\theta_i, p) = E_{w,v} \left[ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x_t^T W x_t + z_t^T C^T(p) W_u C(p) z_t) dt \right].$$

$$J(\theta_i, p) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \{ \text{tr}[W_x E_{w,v}(x_t x_t^T)] + \text{tr}[C^T(p) W_u C(p) E_{w,v}(z_t z_t^T)] \} dt.$$

Let

$$X_t(\theta) = E_{w,v} \begin{pmatrix} x_t x_t^T & x_t z_t^T \\ z_t x_t^T & z_t z_t^T \end{pmatrix} \quad (21)$$

be the covariance matrix of  $x_t$  and  $z_t$ , and let

$$W(p) = \begin{pmatrix} W_x & 0 \\ 0 & C^T(p) W_u C(p) \end{pmatrix}. \quad (22)$$

Then,

$$J(\theta_i, p) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{tr}[W(p) X_t(\theta_i)] dt.$$

This expression can be further reduced [3],

$$J(\theta_i, p) = \text{tr}[W(p)X(\theta_i)] \quad , \quad (23)$$

where  $X(\theta_i)$  for each  $\theta_i$  is the steady state covariance of  $x_t$  and  $z_t$  and is the solution of the algebraic Lyapunov equation:

$$S(\theta_i, p)X(\theta_i) + X(\theta_i)S'(\theta_i, p) + N(p)VN'(p) = 0 \quad . \quad (24)$$

Then,

$$J(p, X) = \sum_{i=1}^N \text{tr}[f_i W(p)X(\theta_i)] \quad . \quad (25)$$

Minimization of Equation (25) by choice of  $p$ , together with  $N$  Equations (24), form an optimization problem with equality constraints.

A Hamiltonian function may be formed:

$$H(p, X, \Lambda) = \sum_{i=1}^N \{ \text{tr}[f_i W(p)X(\theta_i)] + \text{tr}[\Lambda(\theta_i) \{ S(\theta_i, p)X(\theta_i) + X(\theta_i)S'(\theta_i, p) + N(p)VN'(p) \}] \} \quad , \quad (26)$$

where  $\Lambda(\theta_i)$  are  $N$  Lagrange multiplier matrices.

Necessary conditions for minimization of (26) by choice of  $p$  are  $N$  Equations (24) plus  $N$  equations:

$$\frac{\partial H}{\partial X(\theta_i)} = \Lambda(\theta_i)S(\theta_i, p) + S'(\theta_i, p)\Lambda(\theta_i) + f_i W(p) = 0 \quad , \quad (27)$$

and the gradient equations for  $p$ :

$$\begin{aligned} \frac{\partial H}{\partial p_j} = \sum_{i=1}^N \left( \text{tr} \left\{ \left[ f_i \frac{\partial W(p)}{\partial p_j} + \Lambda(\theta_i) \frac{\partial S(\theta_i, p)}{\partial p_j} + \frac{\partial S'(\theta_i, p)}{\partial p_j} \Lambda(\theta_i) \right] X(\theta_i) \right\} \right. \\ \left. + \text{tr} \left\{ \Lambda(\theta_i) \left[ \frac{\partial N(p)}{\partial p_j} VN'(p) + N(p)V \frac{\partial N'(p)}{\partial p_j} \right] \right\} \right) = 0 \quad , \end{aligned} \quad (28)$$

for  $j=1, \dots, \text{DIM}(p)$ .

Note that (24), (27), (28) form a coupled system, with (27) adjoint to (24). However, with a choice of  $p$ , Equations (24) and (27) determine  $X(\theta_1)$  and  $\Lambda(\theta_1)$ , and given  $X(\theta_1)$  and  $\Lambda(\theta_1)$ , Equation (28) determines a value of  $p$  that gives a stationary value of  $J$  in Equation (25).

The existence of a solution for continuous distributions as a limit of the solutions of a series of discrete distributions follows from the definition of the Lebesgue integral and the uniqueness of  $p$ .

The covariance equations and the adjoint Lagrange multiplier equations, (24) and (27), have unique solutions for all  $\theta \in \Theta$ , so that  $J(\theta, p)$  is well defined by (23). Therefore, since (24) and (27) are linear, it is possible to find expressions for  $X(\theta)$  and  $\Lambda(\theta)$  which are continuous in  $\theta$ .

Then,

$$J(\theta, p) = \text{tr}[W(p)X(\theta)] \quad . \quad (29)$$

A Hamiltonian function may be formed,

$$\begin{aligned} H(p, X, \Lambda) = & \int_{\Theta} \text{tr}[W(p)X(\theta)]f(\theta)d\theta \\ & + \int_{\Theta} \text{tr}[\Lambda(\theta)\{S(\theta, p)X(\theta) + X(\theta)S'(\theta, p) + N(p)VN'(p)\}]d\theta \end{aligned} \quad (30)$$

and

$$\begin{aligned} \frac{\partial H}{\partial p_j} = & \int_{\Theta} \text{tr}\left[\frac{\partial W(p)}{\partial p_j} X(\theta)\right]f(\theta)d\theta \\ & + \int_{\Theta} \text{tr}\left\{\left[\Lambda(\theta) \frac{\partial S(\theta, p)}{\partial p_j} + \frac{\partial S'(\theta, p)}{\partial p_j} \Lambda(\theta)\right]X(\theta)\right\}d\theta \\ & + \int_{\Theta} \text{tr}\left\{\Lambda(\theta)\left[\frac{\partial N(p)}{\partial p_j} VN'(p) + N(p)V \frac{\partial N'(p)}{\partial p_j}\right]\right\}d\theta \quad , \end{aligned} \quad (31)$$

for  $j=1, \dots, \text{DIM}(p)$ .

By the properties of the Lebesgue integral [5], (26) and (28) converge to (30) and (31) as  $N \rightarrow \infty$ . Since  $p_N$  is unique for each solution corresponding to a distribution,  $f_N(\theta)$ ,

$$p_N \rightarrow p_c \text{ as } N \rightarrow \infty$$

where  $p_c$  is the solution for the continuous distribution.

It remains to find a procedure for solving the system (24), (25), (27), (28). An iterative procedure may be used. At each step,  $J(p)$  and its gradients,  $\frac{\partial H(p)}{\partial p_i}$ , can be calculated given a distribution,  $f_N(\theta)$ , and a value of  $p$ . Then,  $J(p)$  and its gradients may be used to determine a new value of  $p$  for the next step. These steps are repeated until a value of  $p$  is determined that gives a stationary value for  $J(p)$ . A conjugate gradient algorithm from the IMSL Library is used to perform the iterations. The resulting routine is in the Appendix. See also References [4], [6] and [8].

### 3.4. Examples

The following one-dimensional system will be considered:

$$dx_t = \theta_1 x_t dt + \theta_2 u_t dt + dw_t \quad (32a)$$

$$dy_t = x_t dt + dv_t \quad (32b)$$

$$E \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad E \left[ \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} \begin{pmatrix} dw_t \\ dv_t \end{pmatrix}^T \right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} dt$$

and

$$J(u_t, \theta) = E_{w,v} \left[ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x_t^2 + u_t^2) dt \right] \quad (33)$$

In Example 3-1,  $\theta_2$  will be known and equal to 1,  $\theta_1$  will be uniformly distributed with  $E\theta_1 = 1$ . In Example 3-2,  $\theta_1 = 1$  and  $\theta_2$  will be uniformly distributed with  $E\theta_2 = 1$ .

The uniform distributions are of the form:

$$f_c(\theta) = \begin{cases} \frac{1}{b_c - a_c} & a_c \leq \theta \leq b_c \\ 0 & \text{otherwise} \end{cases} \quad (34a)$$

and

$$E_c(\theta) = 1 = \frac{a_c + b_c}{2} \quad (34b)$$

A series of discrete distributions that converges to  $f_c(\theta)$  is:

$$f_N(\theta) = \begin{cases} \frac{1}{N} & \theta = a_N, a_N + \frac{(b_N - a_N)}{N-1}, a_N + \frac{2(b_N - a_N)}{N-1}, \dots, b_N \\ 0 & \text{otherwise} \end{cases} \quad (35a)$$

and  $a_N, (b_N - a_N)$  are chosen such that

$$\begin{aligned} E_N(\theta) &= E_c(\theta) = 1 \\ \text{VAR}_N(\theta) &= \text{VAR}_c(\theta) \end{aligned} \quad (35b)$$

For the uniform distribution,  $U[a_c, b_c]$ ,

$$\begin{aligned} E_c(\theta) &= \int_{a_c}^{b_c} \frac{1}{b_c - a_c} \theta d\theta \\ E_c(\theta) &= \frac{a_c + b_c}{2} \end{aligned} \quad (36)$$



$$\begin{aligned}\text{VAR}_c(\theta) &= \int_{a_c}^{b_c} \frac{1}{b_c - a_c} \theta^2 d\theta - \left( \frac{a_c + b_c}{2} \right)^2 \\ &= \frac{1}{12} (b_c - a_c)^2.\end{aligned}\quad (37)$$

For the discrete distributions  $f_N(\theta)$ ,

$$\begin{aligned}E_N(\theta) &= \sum_{i=1}^N \frac{1}{N} \theta_i \\ E_N(\theta) &= \frac{a_N + b_N}{2}\end{aligned}\quad (38)$$

$$\text{VAR}_N(\theta) = \sum_{i=1}^N \frac{1}{N} \theta_i^2 - \left( \frac{a_N + b_N}{2} \right)^2$$

$$\begin{aligned}\text{VAR}_N(\theta) &= \frac{1}{N} a_N^2 + \frac{1}{N} \left[ a_N + \frac{(b_N - a_N)}{(N-1)} \right]^2 + \dots + \frac{1}{N} \left[ a_N + \frac{(N-2)(b_N - a_N)}{(N-1)} \right]^2 + \frac{1}{N} b_N^2 \\ &\quad - \left( \frac{a_N + b_N}{2} \right)^2\end{aligned}$$

$$\begin{aligned}\text{VAR}_N(\theta) &= \left( \frac{1}{N} - \frac{1}{4} \right) a_N^2 + \left( \frac{1}{N} - \frac{1}{4} \right) b_N^2 - \frac{1}{2} a_N b_N \\ &\quad + \frac{1}{N} \left[ a_N^2 + \frac{2a_N(b_N - a_N)}{(N-1)} + \frac{(b_N - a_N)^2}{(N-1)^2} \right] \\ &\quad + \frac{1}{N} \left[ a_N^2 + \frac{4a_N(b_N - a_N)}{(N-1)} + \frac{4(b_N - a_N)^2}{(N-1)^2} \right] \\ &\quad + \dots \\ &\quad + \frac{1}{N} \left[ a_N^2 + \frac{2(N-2)a_N(b_N - a_N)}{(N-1)} + \frac{(N-2)^2(b_N - a_N)^2}{(N-1)^2} \right]\end{aligned}$$

$$\begin{aligned}\text{VAR}_N(\theta) &= \left( \frac{1}{N} - \frac{1}{4} \right) a_N^2 + \left( \frac{1}{N} - \frac{1}{4} \right) b_N^2 - \frac{1}{2} a_N b_N \\ &\quad + \frac{(N-2)}{N} a_N^2 +\end{aligned}$$

$$\begin{aligned}
& + \frac{2[1 + 2 + \dots + (N-2)]a_N(b_N - a_N)}{N(N-1)} \\
& + \frac{[1^2 + 2^2 + \dots + (N-2)^2](b_N - a_N)^2}{N(N-1)}
\end{aligned}$$

Let

$$\begin{aligned}
S_0 &= 0 \\
S_N &= S_{N-1} + N^2
\end{aligned} \tag{39}$$

Then for  $N \geq 2$ ,  $[1^2 + 2^2 + \dots + (N-2)^2] = S_{N-2}$ . Also,

$$[1 + 2 + \dots + (N-2)] = \frac{(N-2)(N-1)}{2}$$

Therefore,

$$\begin{aligned}
\text{VAR}_N(\theta) &= \left(\frac{1}{N} - \frac{1}{4}\right) a_N^2 + \left(\frac{1}{N} - \frac{1}{4}\right) b_N^2 - \frac{1}{2} a_N b_N \\
&+ \frac{(N-2)}{N} a_N b_N + \frac{S_{N-2}}{N(N-1)^2} (b_N - a_N)^2 \\
\text{VAR}_N(\theta) &= \left[ \frac{1}{N} - \frac{1}{4} + \frac{S_{N-2}}{N(N-1)^2} \right] (b_N - a_N)^2
\end{aligned} \tag{40}$$

Then using Equations (35b)

$$(b_N - a_N) = \frac{(b_c - a_c)}{\sqrt{12 \left( \frac{1}{N} - \frac{1}{4} + \frac{S_{N-2}}{N(N-1)^2} \right)}} \tag{41a}$$

$$a_N = 1 - \frac{(b_N - a_N)}{2} \tag{41b}$$

Given a continuous distribution, (34), Equations (35a), (39), (41) describe the series of discrete distributions converging to  $f_c(\theta)$ .

The block diagonal form for first and second-order controllers are:

1st order -

$$\begin{aligned} u_t &= z_t \\ dz_t &= p_1 z_t dt + p_2 dy_t \end{aligned} \quad (42)$$

The closed loop system is

$$\begin{pmatrix} dx_t \\ dz_t \end{pmatrix} = \begin{pmatrix} \theta_1 & \theta_2 \\ p_2 & p_1 \end{pmatrix} \begin{pmatrix} x_t \\ z_t \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} \quad (43)$$

2nd order -

$$\begin{aligned} u_t &= (0 \quad 1) \begin{pmatrix} z_{1t} \\ z_{2t} \end{pmatrix} \\ \begin{pmatrix} dz_{1t} \\ dz_{2t} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} z_{1t} \\ z_{2t} \end{pmatrix} dt + \begin{pmatrix} p_3 \\ p_4 \end{pmatrix} dy_t \end{aligned} \quad (44)$$

The closed loop system is

$$\begin{pmatrix} dx_t \\ dz_{1t} \\ dz_{2t} \end{pmatrix} = \begin{pmatrix} \theta_1 & 0 & \theta_2 \\ p_3 & 0 & 1 \\ p_4 & p_1 & p_2 \end{pmatrix} \begin{pmatrix} x_t \\ z_{1t} \\ z_{2t} \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & p_3 \\ 0 & p_4 \end{pmatrix} \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} \quad (45)$$

In Example 3-1 we set  $\theta_2 = 1$  and consider  $\theta_1$  unknown. Let

$$X(\theta) = \begin{pmatrix} X_1 & X_2 \\ X_2 & X_3 \end{pmatrix}, \quad \Lambda(\theta) = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2 & \Lambda_3 \end{pmatrix}$$

Then Equations (24), (27), (28) reduce to:

(24):

$$\begin{aligned} 2\theta_1 X_1 + 2X_2 + 1 &= 0 \\ p_2 X_1 + (\theta_1 + p_2) X_2 + X_3 &= 0 \\ 2p_2 X_2 + 2p_1 X_3 + p_2^2 &= 0 \end{aligned}$$

(27):

$$2\theta_1\Lambda_1 + 2p_2\Lambda_2 + f_1 = 0$$

$$\Lambda_1 + (\theta_1 + p_1)\Lambda_2 + p_2\Lambda_3 = 0$$

$$2\Lambda_2 + 2p_1\Lambda_3 + f_1 = 0$$

(28):

$$\frac{\partial H}{\partial p_1} = 2\Lambda_2 X_2 + 2\Lambda_3 X_3 = 0$$

$$\frac{\partial H}{\partial p_2} = 2\Lambda_2 X_1 + 2\Lambda_3 X_2 + 4\Lambda_3 p_2^3 = 0$$

The gradient method discussed earlier will be used to solve these equations. However, since the procedure is a local one, an adequate starting point must be found. One such point that has proven useful is the optimal regulator solution for  $\theta_1 = E\theta_1 = 1$ . This can be found by solving the well-known optimal regulator equations and transforming them into the block diagonal coordinates. The optimal regulator for  $\theta_1 = 1$  is described by [3]:

$$u_t = -P\hat{x}_t \quad (46a)$$

$$d\hat{x}_t = Kx_t dt + (1 - P - K)\hat{x}_t dt + Kdv_t \quad (46b)$$

$$2P - P^2 + 1 = 0, \quad P = 1 + \sqrt{2} = 2.414 \quad (46c)$$

$$2K - K^2 + 1 = 0, \quad K = 1 + \sqrt{2} = 2.414 \quad (46d)$$

But, in block diagonal coordinates,

$$u_t = z_t$$

So the transformation,  $\hat{x}_t = -\frac{1}{P} z_t$  is used, and

$$dz_t = -PKx_t dt + (1 - P - K)z_t dt - PKdv_t$$

Comparing with Equation (43),

$$dz_t = p_2 x_t dt + p_1 z_t dt + p_2 dv_t ,$$

we find

$$p_1 = - (1 - P - K)$$

$$p_2 = - PK .$$

With  $P = 2.414$  and  $K = 2.414$ , the starting point is calculated as,

$$p_1 = - 3.828$$

$$p_2 = - 5.828 .$$

Optimal values for three uniform distributions are calculated. The solutions for some approximating discrete distributions are also shown.

U[0.9,1.1]      VAR $\theta$  = 0.00333

N	$p_1$	$p_2$	J
3	-3.8275	-5.8915	16.6942
7	-3.8273	-5.8917	16.6948
15	-3.8273	-5.8917	16.6949
continuous	-3.8273	-5.8917	16.6949

U[0.7,1.3]      VAR = 0.03000

N	$p_1$	$p_2$	J
3	-3.8505	-6.3474	18.2374
7	-3.8453	-6.3577	18.2717
15	-3.8443	-6.3595	18.2776
continuous	-3.8441	-6.3600	18.2792

U[0.5,1.5]      VAR $\theta$  = 0.08333

N	$p_1$	$p_2$	J
3	-3.9481	-7.1189	20.9582
7	-3.9333	-7.1679	21.1293
15	-3.9299	-7.1767	21.1605
continuous	-3.9289	-7.1791	21.1693 .

Note that when the variance is small, convergence to the continuous solution is very rapid, as can be seen especially in the U[0.9,1.1] distribution, where four significant figures are achieved at  $N=3$ . However, the

convergence rate decreases as the variance increases, so that for the  $U[0.7,1.3]$  distribution four significant figures are not achieved until  $N=15$ , while for the  $U[0.5,1.5]$  distribution even more exact distributions are required for the same precision.

Once optimal parameters for a particular distribution are found, it is interesting to see how the actual cost varies as a function of  $\theta$ . It is desired that the cost incurred for a particular  $\theta$  be as close as possible to the cost incurred if the optimal regulator was used for that value of  $\theta$ . Therefore, defining  $J_{LS}(\theta)$  as the cost incurred by the least sensitive controller, and  $J_{OPT}(\theta)$  as the cost incurred by an optimal regulator designed for  $\theta$ , a performance measure that may be used is a plot of  $J_{LS}(\theta) - J_{OPT}(\theta)$  versus  $\theta$ . Some interesting properties of these systems can be discovered.

Figures 1 to 3 show  $J_{LS}(\theta) - J_{OPT}(\theta)$  for the three distributions considered. From these curves, several properties of the least sensitive solutions can be discerned.

- 1) As can be expected, the relative cost increases as the true value of  $\theta$  deviates from the mean. This is a desirable property since less performance is sacrificed as the true value of  $\theta$  moves toward the nominal value.

- 2)  $J_{OPT}(\theta)$  is not achieved at any  $\theta$ . If  $J_{OPT}(\theta)$  was achieved at a value of  $\theta$ , it would be an optimal regulator for that  $\theta$ . Therefore, the least sensitive solutions are not optimal for any  $\theta$ . Also, this fact shows that disturbance attenuation is always sacrificed to achieve less sensitivity.

- 3) More disturbance attenuation is sacrificed as less sensitivity is desired. This property may be made clearer by investigating the performance at the mean:

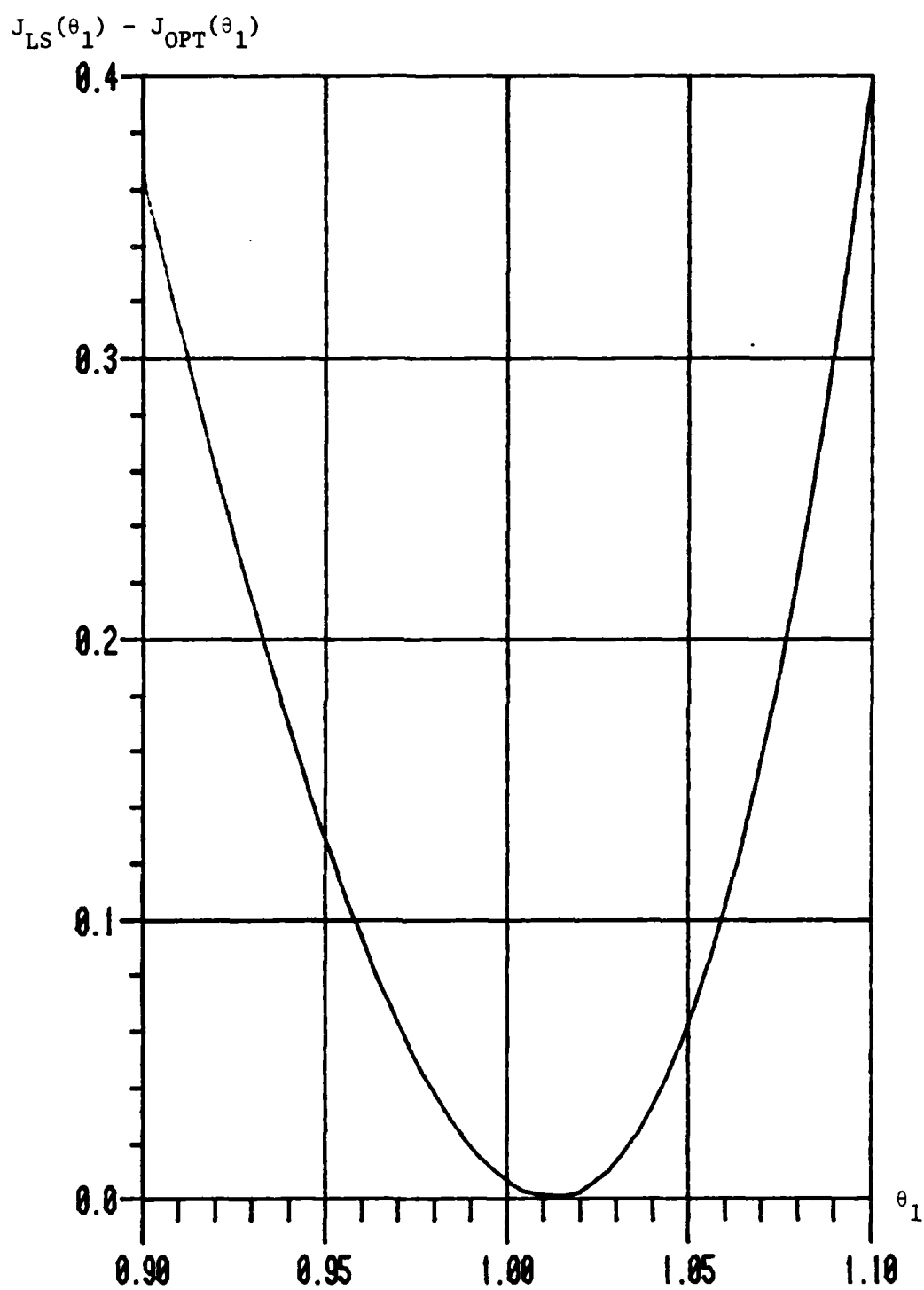


Figure 1.  $J_{LS} - J_{OPT}$  vs.  $\theta_1$ ,  $\theta_2=1$ ,  $\theta_1 \sim U[0.9, 1.1]$ .

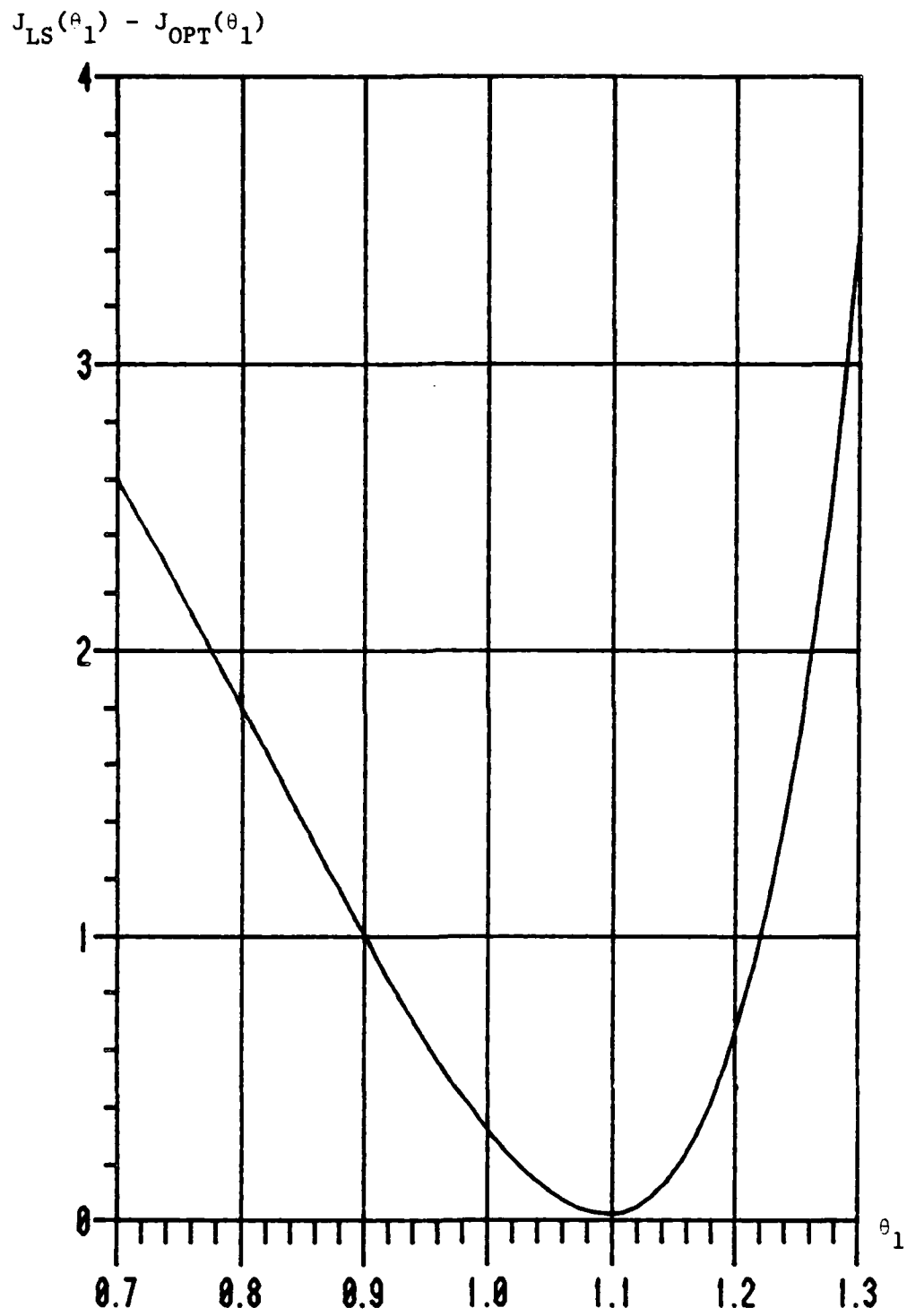


Figure 2.  $J_{LS} - J_{OPT}$  vs.  $\theta_1$ ,  $\theta_2=1$ ,  $\theta_1 \sim U[0.7, 1.3]$ .



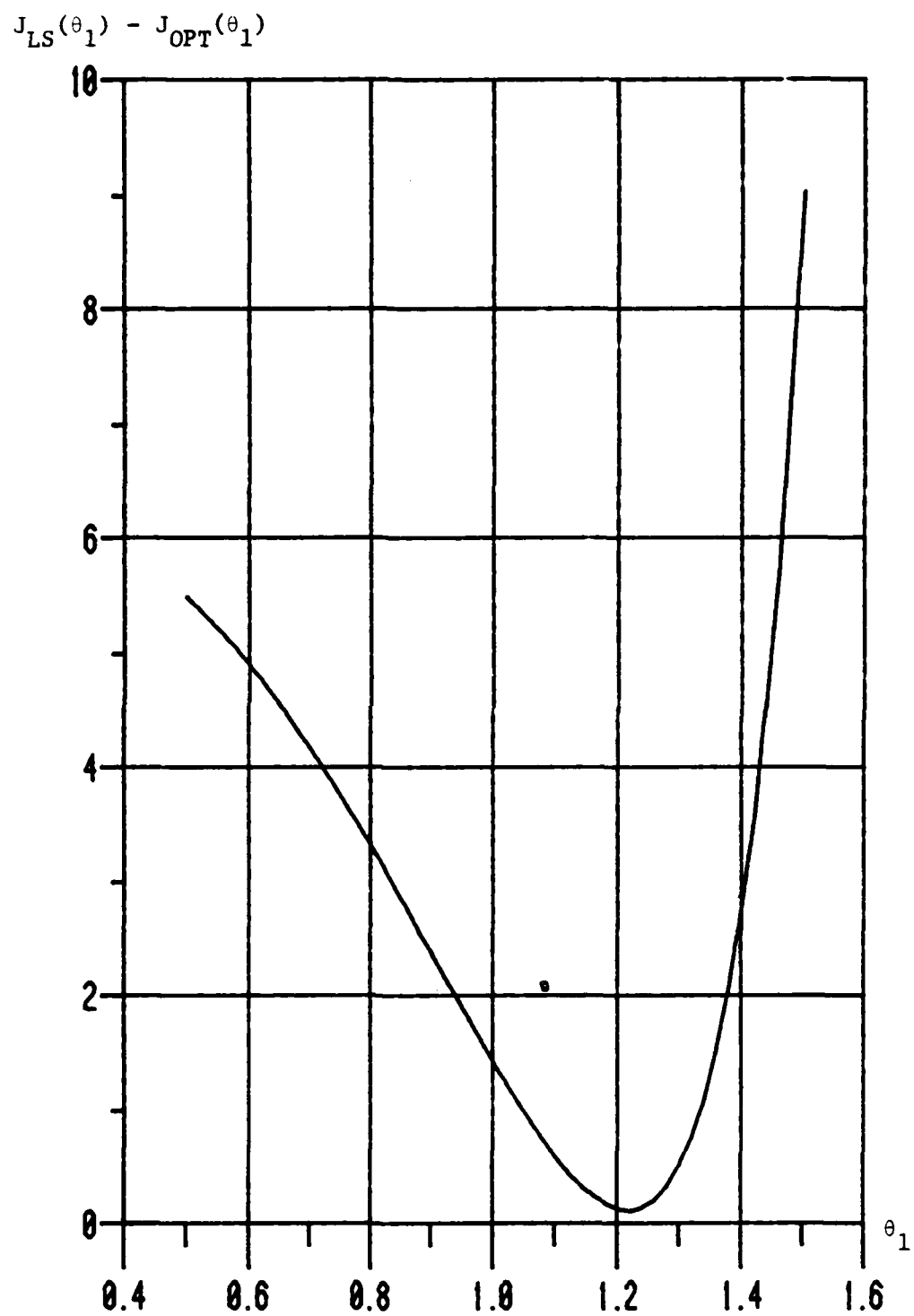


Figure 3.  $J_{LS} - J_{OPT}$  vs.  $\theta_1$ ,  $\theta_2=1$ ,  $\theta_1 \sim U[0.5, 1.5]$ .

Distribution	VAR $\theta$	$J_{LS}(1.0) - J_{OPT}(1.0)$
U[0.9,1.1]	0.00333	0.03
U[0.7,1.3]	0.03000	0.31
U[0.5,1.5]	0.08333	1.39

As can be seen, performance at the nominal value of  $\theta$  deteriorates as the variance of  $\theta$  increases.

4) The least sensitive controller is biased in the sense that the minimum value of the relative cost occurs at a value of  $\theta$  that is higher than the nominal value. This bias may be quantified by calculating the value of  $\theta$  at the minimum as a percentage of the full deviation possible.

Distribution	Bias	Full Deviation
U[0.9,1.1]	15% above	0.1
U[0.7,1.3]	33% above	0.3
U[0.5,1.5]	45% above	0.5

Figure 4 shows  $J_{OPT}(\theta)$  as a function of  $\theta$ . As can be seen from this figure, the controller is biased toward  $\theta$ 's that are associated with higher costs. Also, the percent bias increases because  $J_{OPT}(\theta)$  increases at an increasing rate.

Second-order controllers may be calculated for this example using the same algorithm. To calculate a starting point, a comparison is made between the transfer functions of the first and second-order controllers:

$$\text{first order: } \frac{U(s)}{Y(s)} = \frac{p_2}{s - p_1}$$

$$\text{second order: } \frac{U(s)}{Y(s)} = \frac{p_4(s + \frac{p_3 p_1}{p_4})}{(s - p_1) \left[ s + p_1 - p_2 + \frac{p_1^2 - p_1 p_2 + p_1}{s + p_1 - p_2} \right]}$$

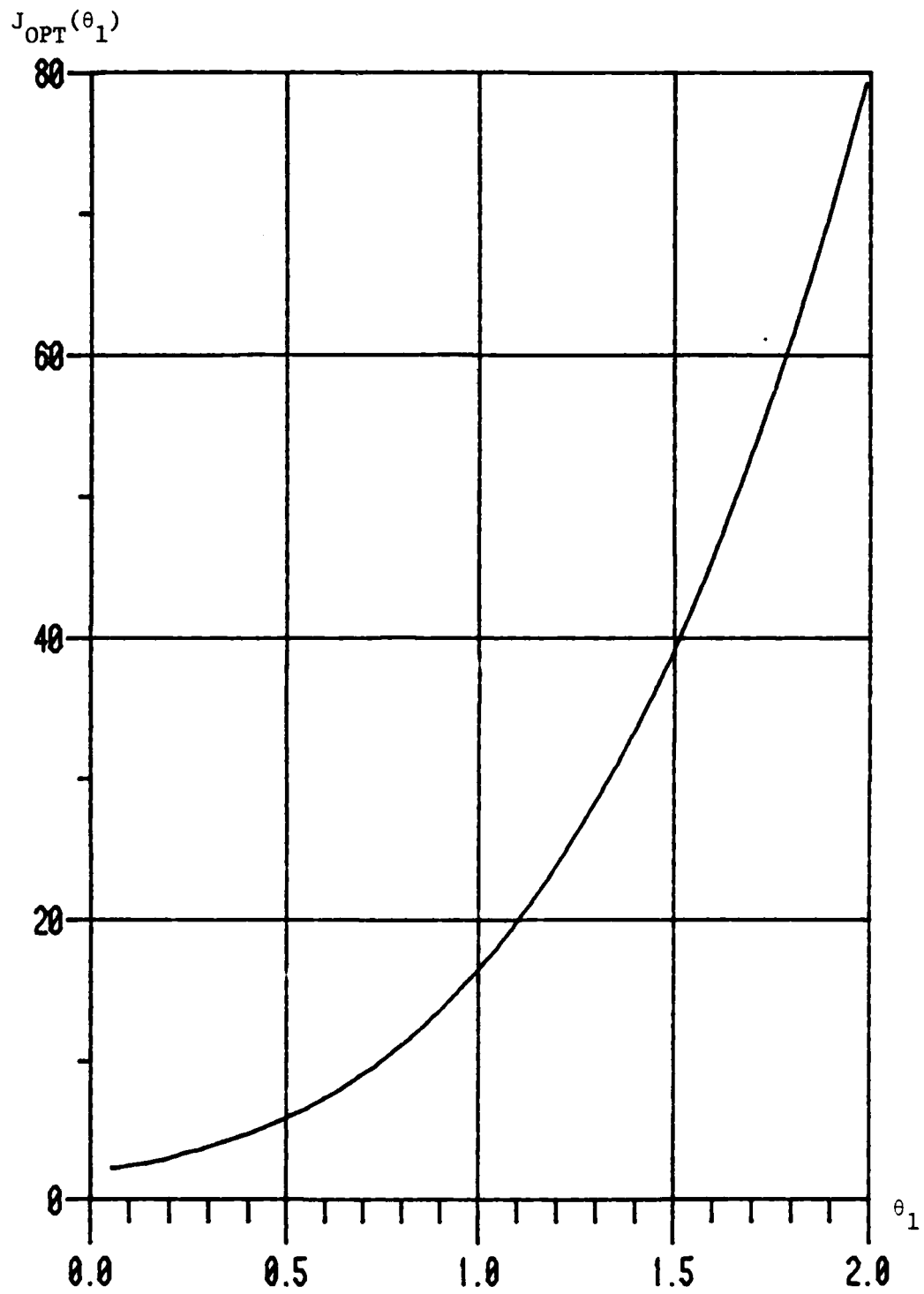


Figure 4.  $J_{OPT}$  vs.  $\theta_1$ ,  $\theta_2=1$ .

Using the starting point for the first-order controller, a starting point for the second-order controller can be formed by matching the two transfer functions. Let

$$p_1 \text{ (second order)} = p_1 \text{ (first order)}$$

$$p_4 \text{ (second order)} = p_2 \text{ (first order)} .$$

Then,  $p_2$  and  $p_3$  can be chosen so that the second pole is stable, and nearly cancels the zero. Therefore, using the starting point for first-order controllers:

$$p_1 = - 3.828$$

$$p_2 = - 5.828$$

set

$$p_1 = - 3.828$$

$$p_4 = - 5.828 .$$

Let  $p_1 - p_2 = 1$ ,

then  $p_2 = p_1 - 1 = - 4.828 .$

Then, so that  $s + p_1 - p_2$  cancels the zero,  $s + \frac{p_3 p_1}{p_4}$ , set

$$\frac{p_3 p_1}{p_4} = p_1 - p_2 = 1$$

$$p_3 = \frac{p_4}{p_1} = 1.522 .$$

Therefore, the starting point for the second-order controllers is:

$$p_1 = - 3.828$$

$$p_2 = - 4.828$$

$$p_3 = 1.522$$

$$p_4 = - 5.828 .$$

Second-order controllers were calculated for the same distributions. The results are summarized below.

U[0.9,1.1]	P <sub>1</sub>	P <sub>2</sub>	P <sub>3</sub>	P <sub>4</sub>	J
1st order	-3.8273	-5.8917			16.6949
2nd order	-4.4948	-5.0194	1.5411	-5.9062	16.6948
U[0.7,1.3]	P <sub>1</sub>	P <sub>2</sub>	P <sub>3</sub>	P <sub>4</sub>	J
1st order	-3.8441	-6.3600			18.2776
2nd order	-5.5380	-5.3948	1.6626	-6.4557	18.2722
U[0.5,1.5]	P <sub>1</sub>	P <sub>2</sub>	P <sub>3</sub>	P <sub>4</sub>	J
1st order	-3.9289	-7.1791			21.1693
2nd order	-7.4618	-6.0202	1.8371	-7.3618	21.1397

The percent improvement in J may be calculated:

Distribution	%ΔJ
U[0.9,1.1]	0.0006%
U[0.7,1.3]	0.0295%
U[0.5,1.5]	0.1398%

These figures show that the improvement found using the second-order controllers is very low, but the improvement increases as the variance of  $\theta$  increases. Figures 5, 6 and 7 show a plot of  $J_{LS}$  (first order) -  $J_{LS}$  (second order) as a function of  $\theta$  for the three distributions. These curves show that this second-order controller does not make improvement over the entire range of  $\theta$ . However, it can be seen that the improvement occurs for  $\theta$ 's that are associated with higher costs.

In Example 3-2,  $\theta_1 = 1$  and  $\theta_2$  is considered to be unknown with  $E\theta_2 = 1$ . Using the same method as in Example 3-1, with the same starting points, optimal values for first and second-order controllers are calculated.

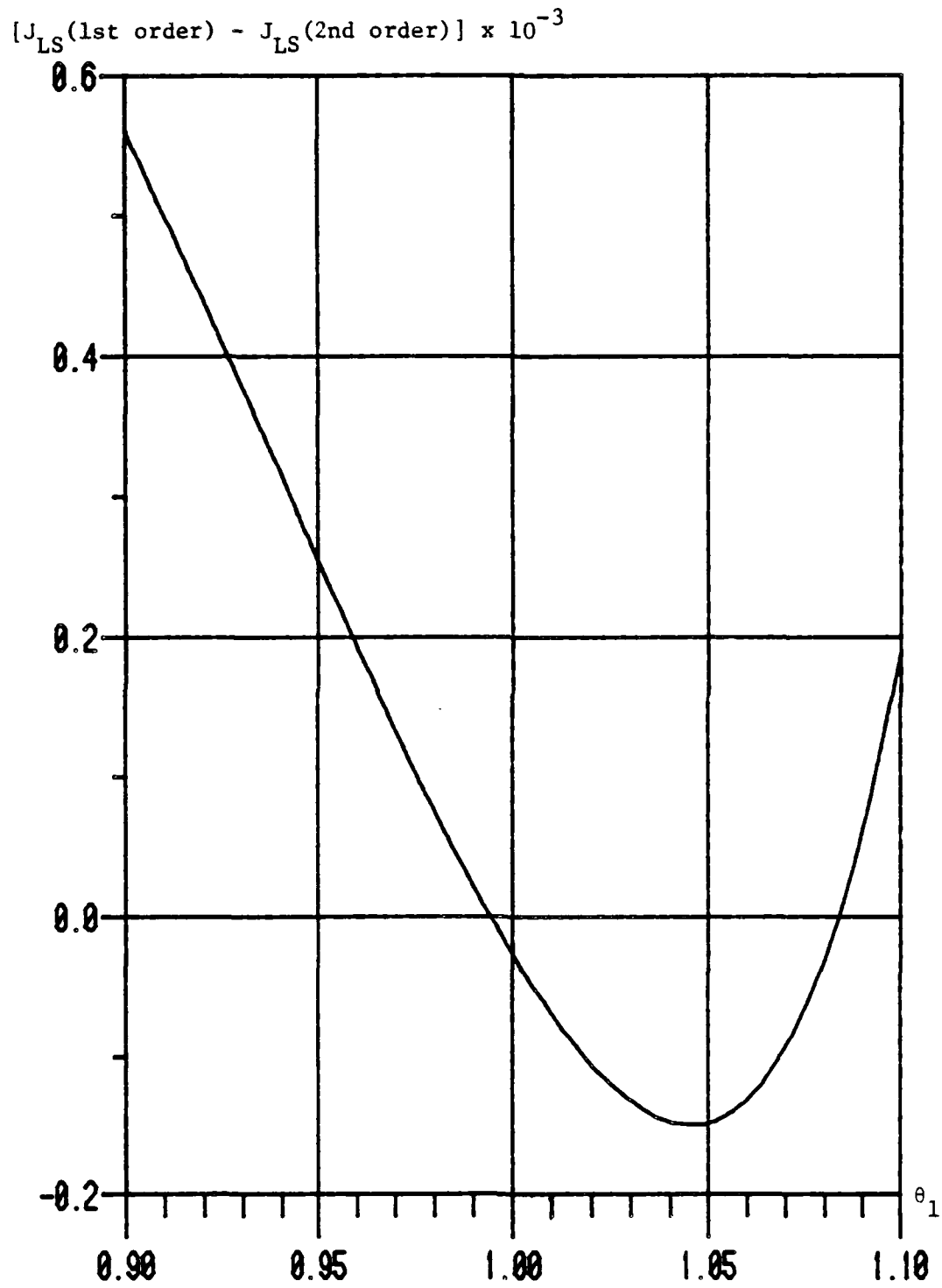


Figure 5.  $J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$  vs.  $\theta_1$ ,  $\theta_2=1$ ,  $\theta_1 \sim U[0.9, 1.1]$ .

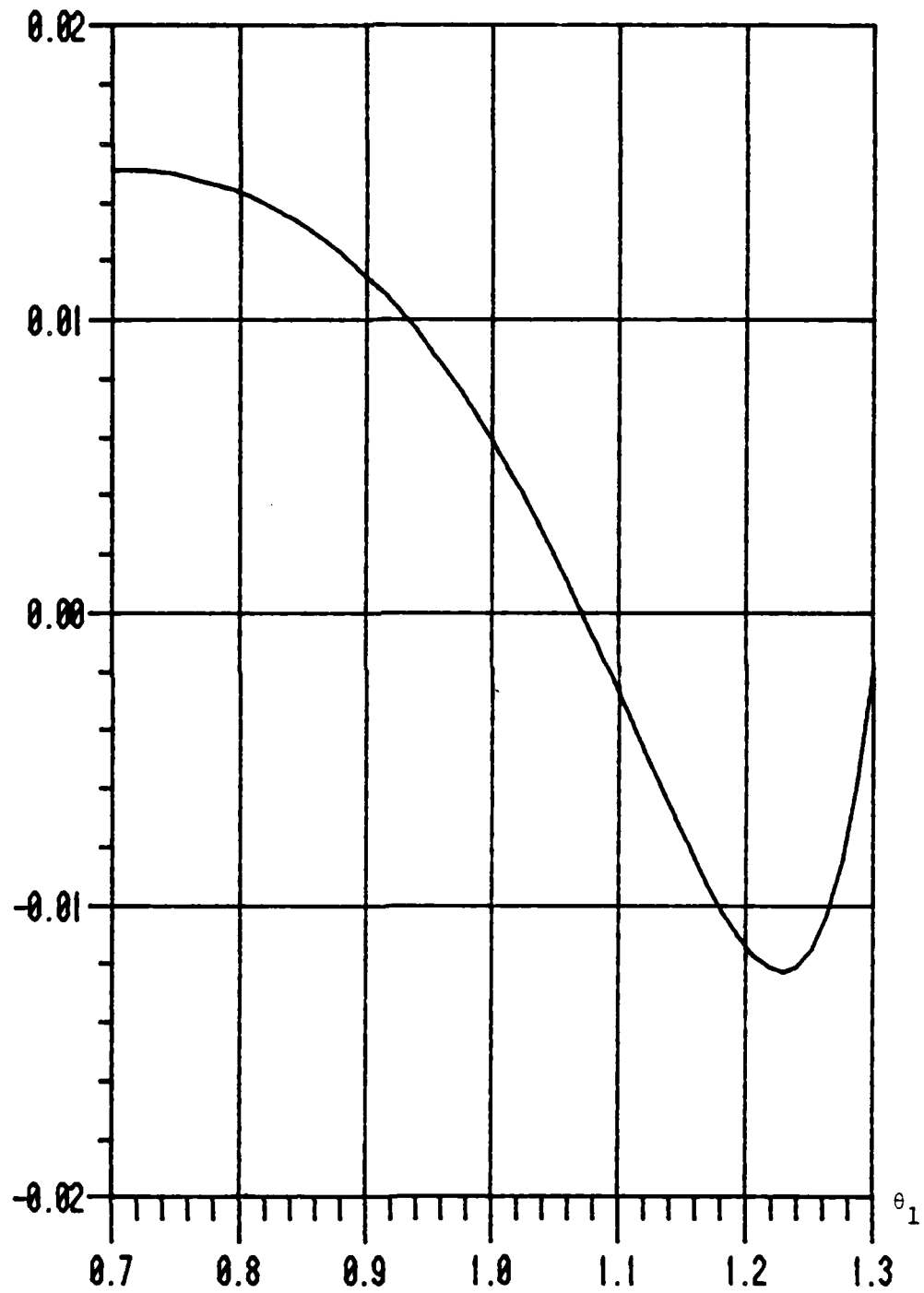
$J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$ 

Figure 6.  $J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$  vs.  $\theta_1$ ,  $\theta_2=1$ ,  $\theta_1 \sim U[0.7, 1.3]$ .

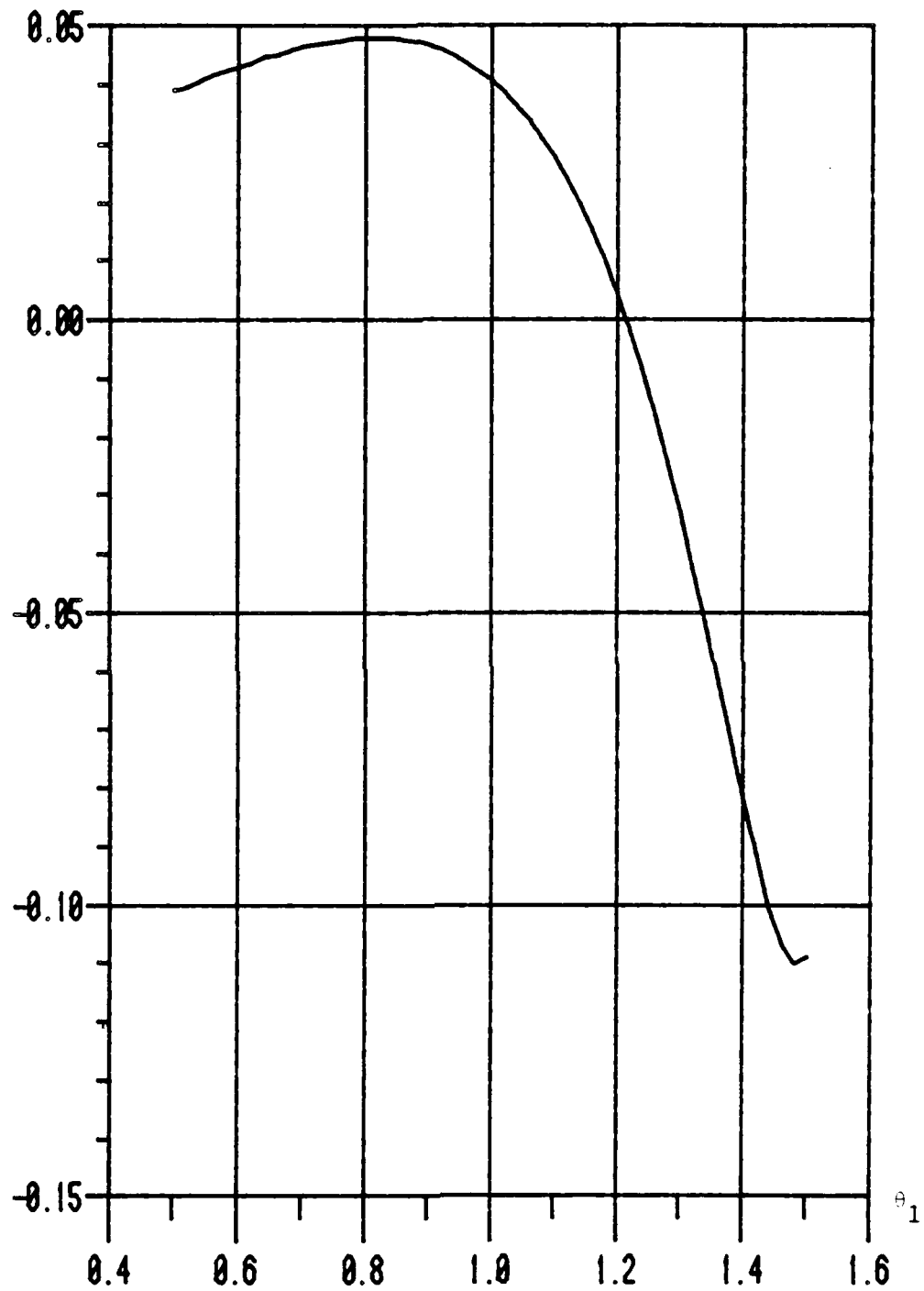
$J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$ 

Figure 7.  $J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$  vs.  $\theta_1$ ,  $\theta_2=1$ ,  $\theta_1 \sim U[0.5, 1.5]$ .



U[0.9,1.1]      VAR $\theta$  = 0.00333

	P <sub>1</sub>	P <sub>2</sub>	P <sub>3</sub>	P <sub>4</sub>	J
first order	-3.8031	-5.8707			16.7323
second order	-3.4583	-4.7587	1.5493	-5.9083	16.7314

U[0.7,1.3]      VAR $\theta$  = 0.03000

	P <sub>1</sub>	P <sub>2</sub>	P <sub>3</sub>	P <sub>4</sub>	J
first order	-3.6087	-6.2470			19.0127
second order	-4.1535	-5.1693	1.7693	-6.5897	18.9244

U[0.5,1.5]      VAR $\theta$  = 0.08333

	P <sub>1</sub>	P <sub>2</sub>	P <sub>3</sub>	P <sub>4</sub>	J
first order	-3.2658	-7.2651			25.8909
second order	-5.6600	-6.1005	2.2744	-8.2686	24.9534

Figures 8, 9 and 10 show plots of  $J_{LS}(\theta) - J_{OPT}(\theta)$  versus  $\theta$  for the first-order controllers. Basically, the conclusions here are the same as in Example 3.1, and only a few differences will be noted here.

1) The relative cost for the parameter in the control gain is, in general, higher than the relative cost for the parameter in the plant dynamics.

2) These controllers are biased below the mean. To quantify:

Distribution	Bias	Full Deviation
U[0.9,1.1]	15% below	0.1
U[0.7,1.3]	42% below	0.3
U[0.5,1.5]	66% below	0.5

Notice also that these biases are greater than in Example 3.1.

Figure 11 shows that  $J_{OPT}(\theta)$  is greatest for parameter values less than the mean, which shows why the biases are below the mean. Also note that

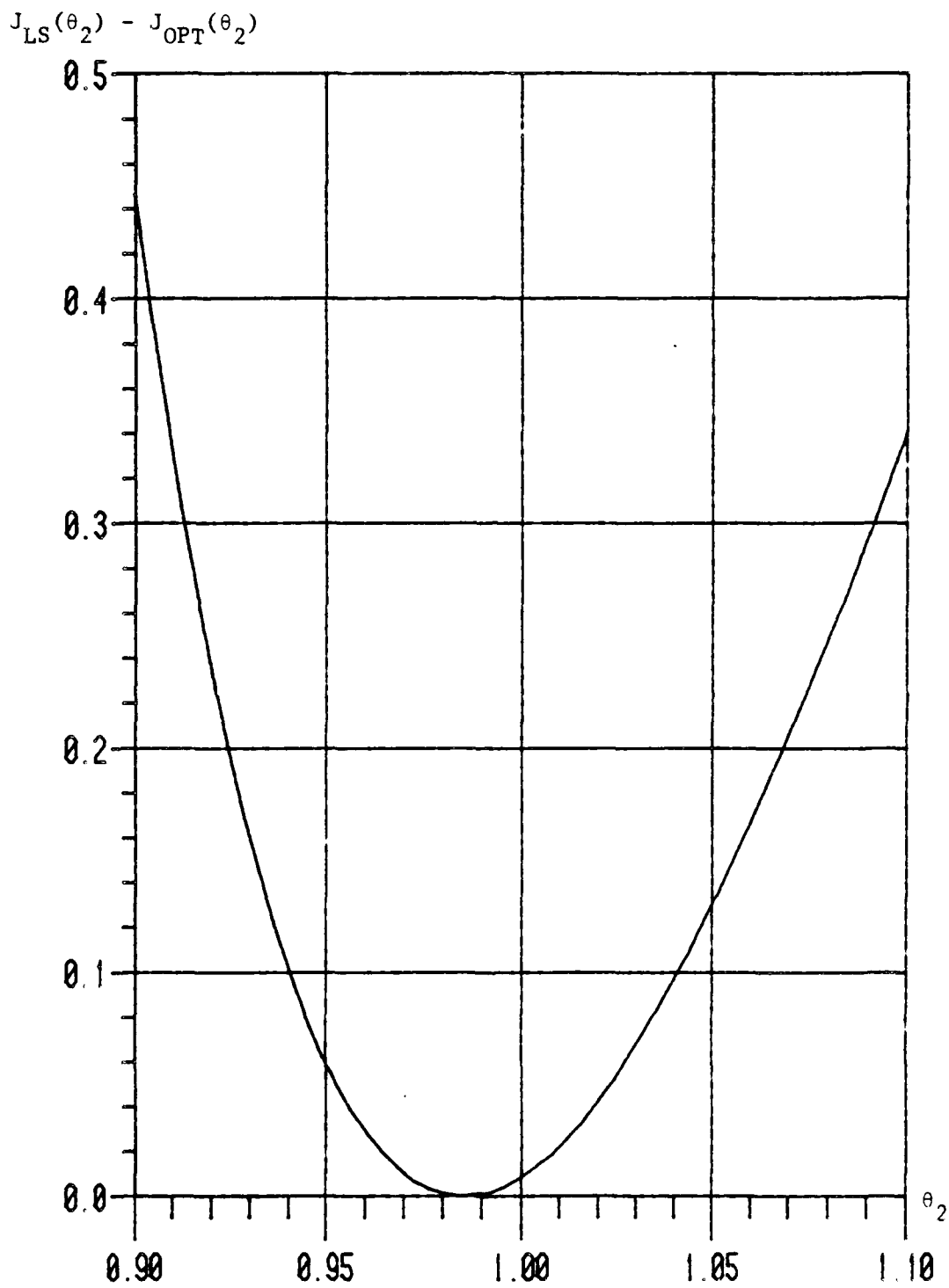


Figure 8.  $J_{LS} - J_{OPT}$  vs.  $\theta_2$ ,  $\alpha_1=1$ ,  $\theta_2 \sim U[0.9, 1.1]$ .

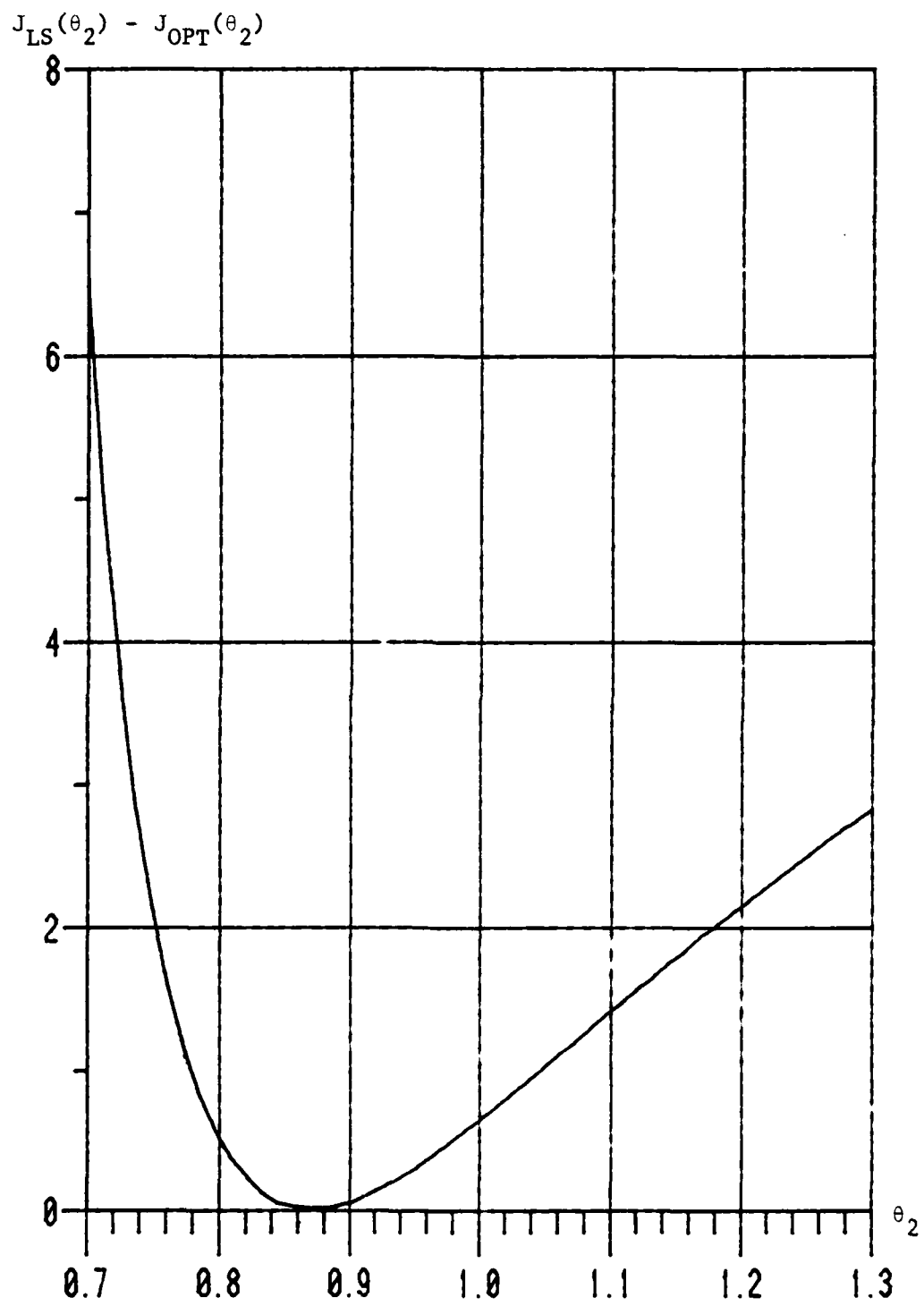


Figure 9.  $J_{LS} - J_{OPT}$  vs.  $\theta_2$ ,  $\theta_1=1$ ,  $\theta_2 \sim U[0.7, 1.3]$ .

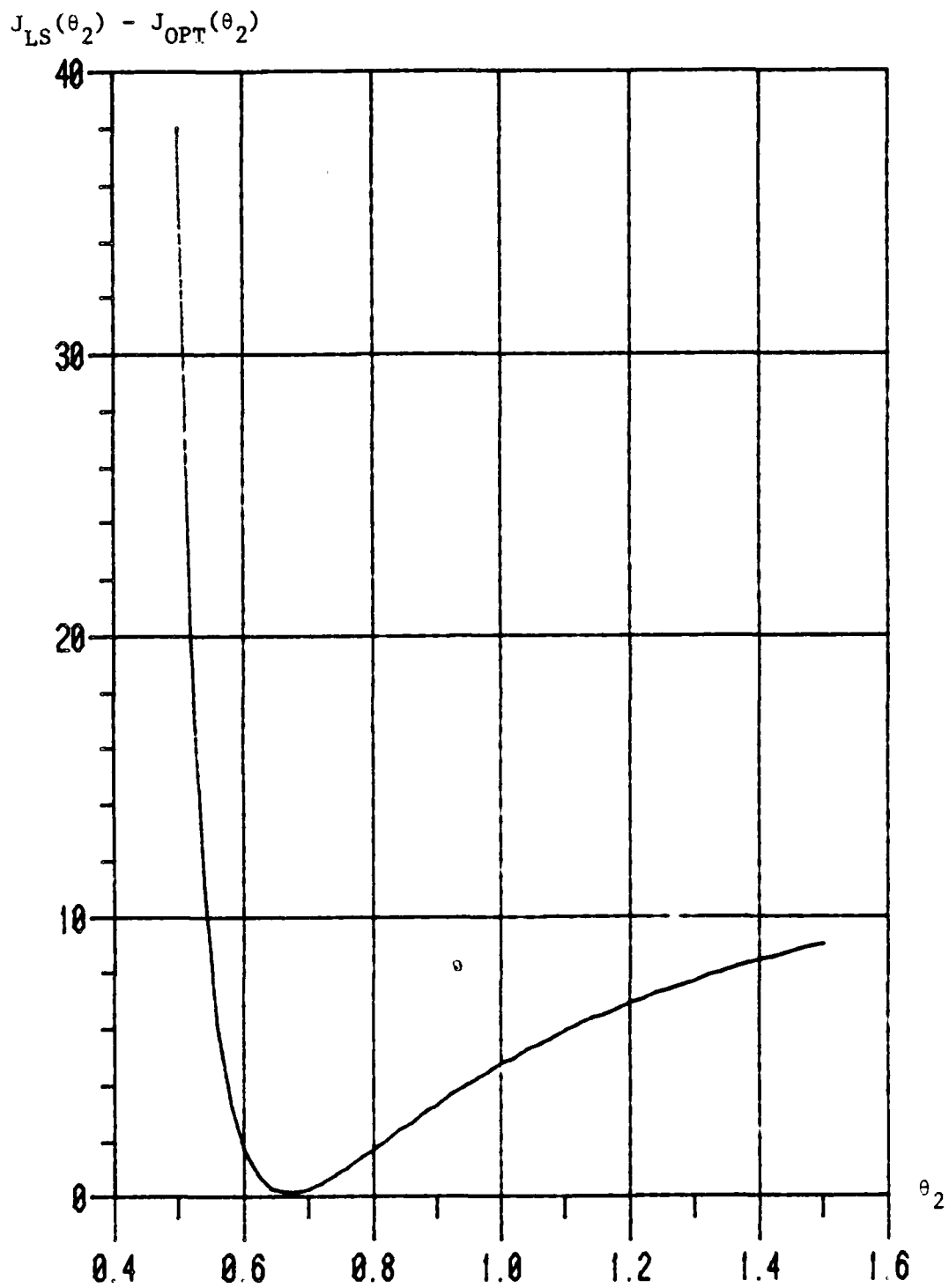


Figure 10.  $J_{LS} - J_{OPT}$  vs.  $\theta_2$ ,  $\theta_1=1$ ,  $\theta_2 \sim U[0.5, 1.5]$ .

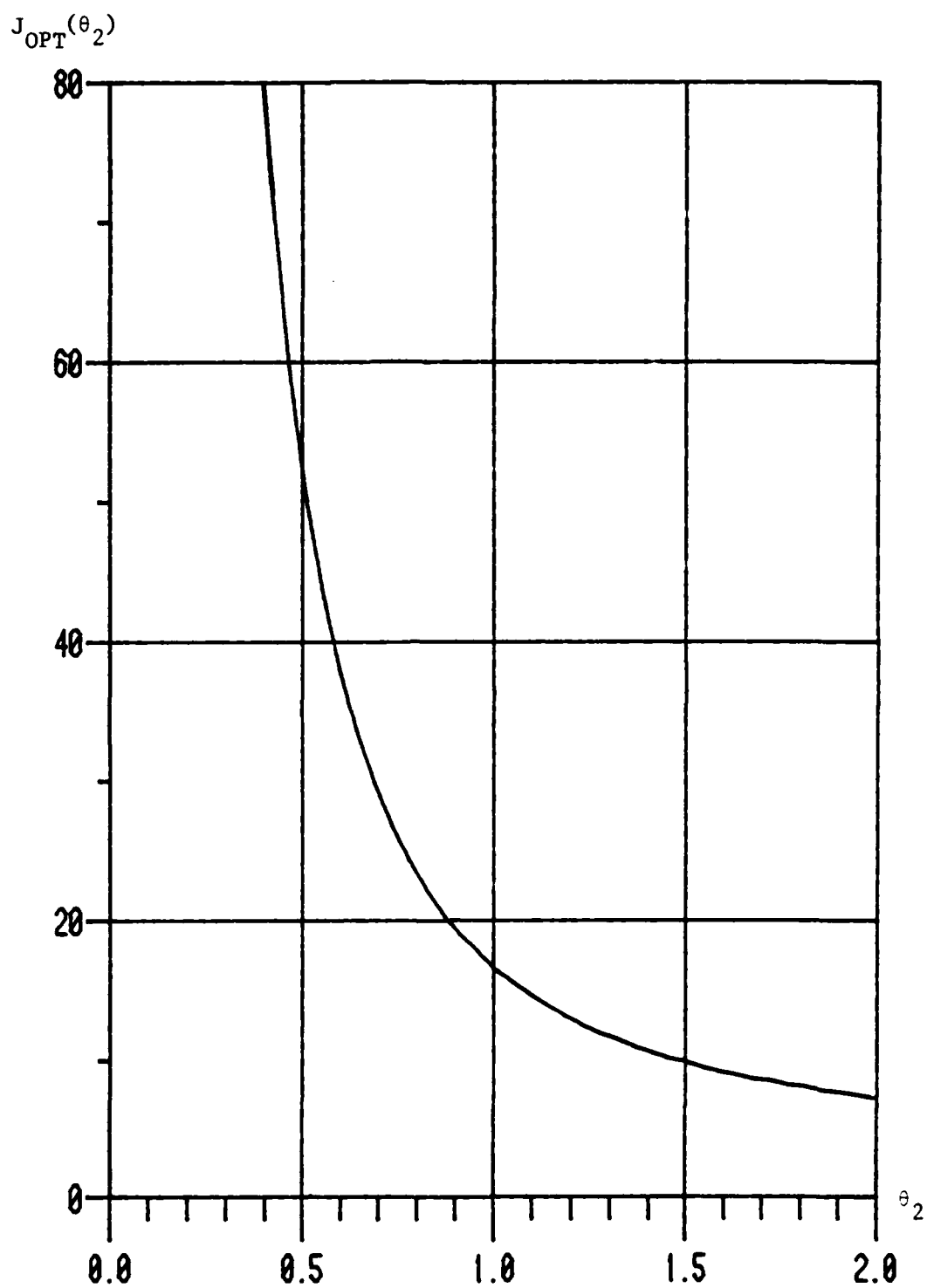


Figure 11.  $J_{OPT}$  vs.  $\theta_2$ ,  $\theta_1=1$ .

$J_{OPT}(\theta)$  is generally greater than in Example 3.1, and increases at a faster rate than in Example 3.1, thereby explaining why the relative costs and biases are greater than those in Example 3.1.

For second-order controllers, the degree of improvement is also greater than that of Example 3.1.

Distribution	$\% \Delta J_{LS}$ (Ex. 1)	$\% \Delta J_{LS}$ (Ex. 2)
U[0.9,1.1]	0.0006%	0.0053%
U[0.7,1.3]	0.0295%	0.4644%
U[0.5,1.5]	0.1398%	3.621%

Note that in this case, the improvement can become significant although still not very great.

Figures 12, 13 and 14 show  $J_{LS}$  (first order) -  $J_{LS}$  (second order) for this example. As seen before, a savings is not made over the entire parameter range, but the improvement is made for  $\theta$ 's associated with higher costs.

### 3.5. Conclusions

To conclude, it has been shown how the method of Ashkenzai and Bryson may be used to find least sensitive controllers for systems with parameters described by continuous distributions. The rate of convergence of the solution is very rapid although it slows down as less parameter sensitivity is desired.

The performance of the least sensitive controllers was discussed. It was concluded that the performance of the system decreased as the distance

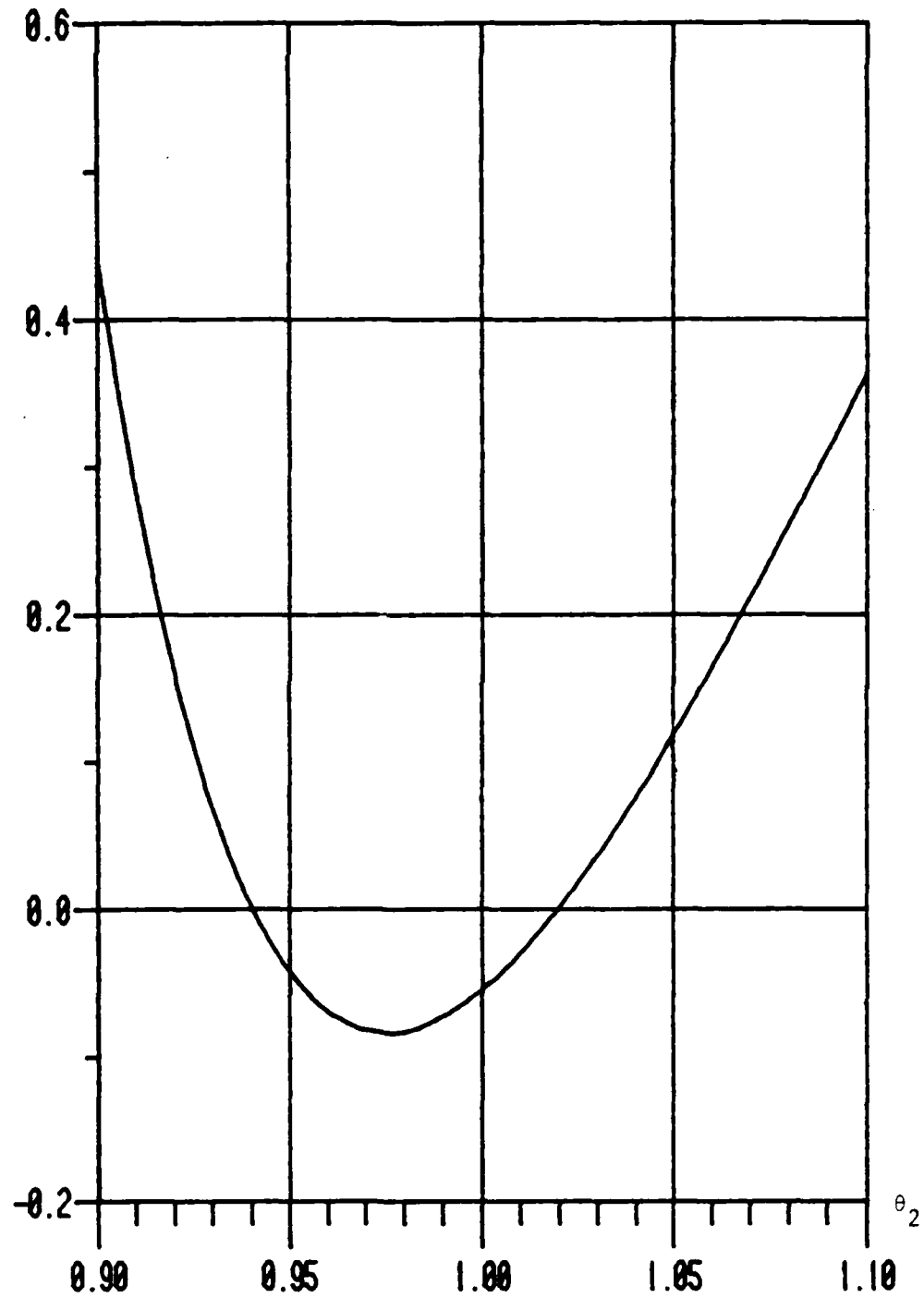
$[J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})] \times 10^{-2}$ 

Figure 12.  $J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$  vs.  $\theta_2$ ,  $\theta_1=1$ ,  
 $\theta_2 \sim U[0.9, 1.1]$ .

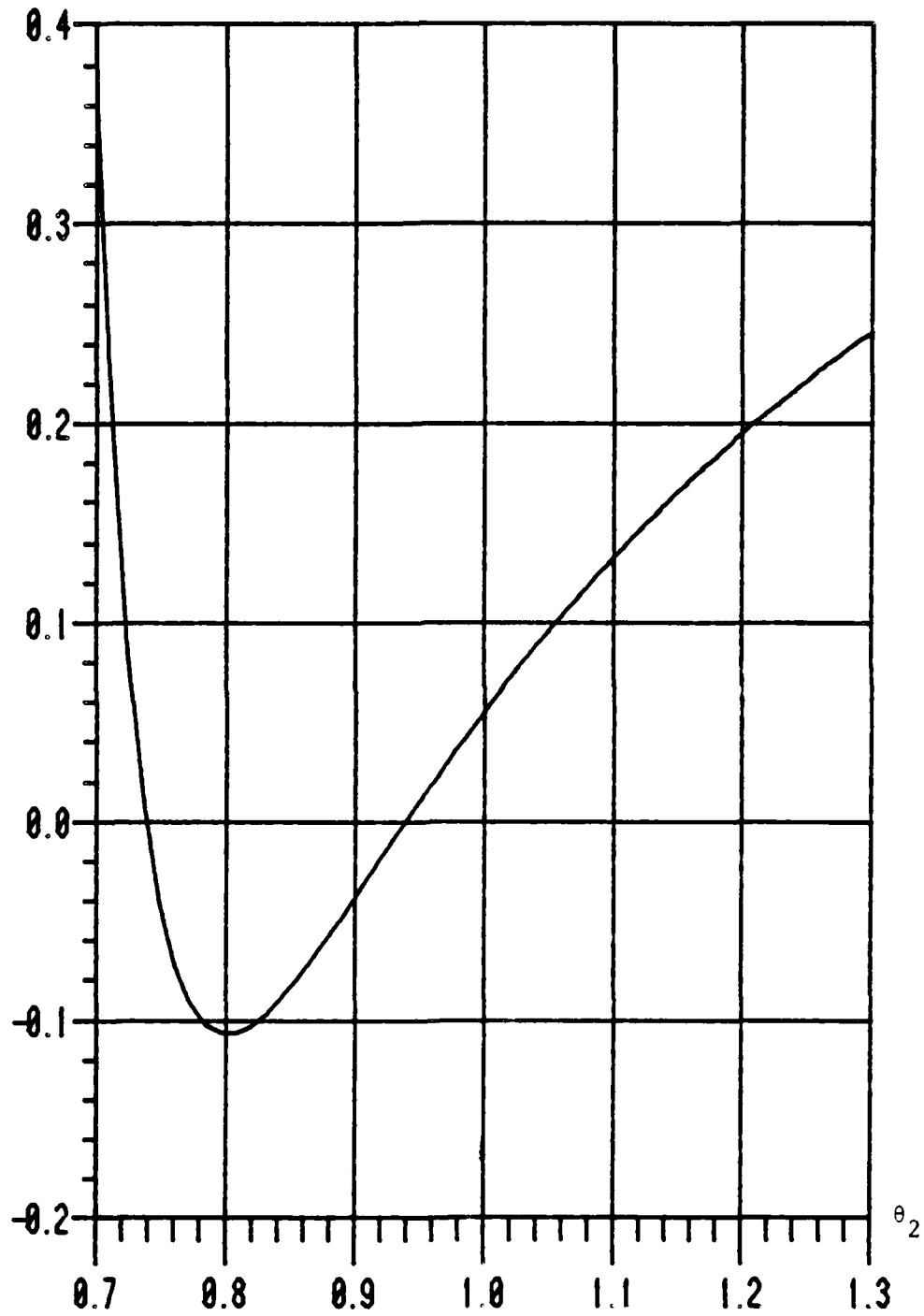
$J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$ 

Figure 13.  $J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$  vs.  $\theta_2$ ,  $\theta_1=1$ ,  $\theta_2 \sim U[0.7, 1.3]$ .



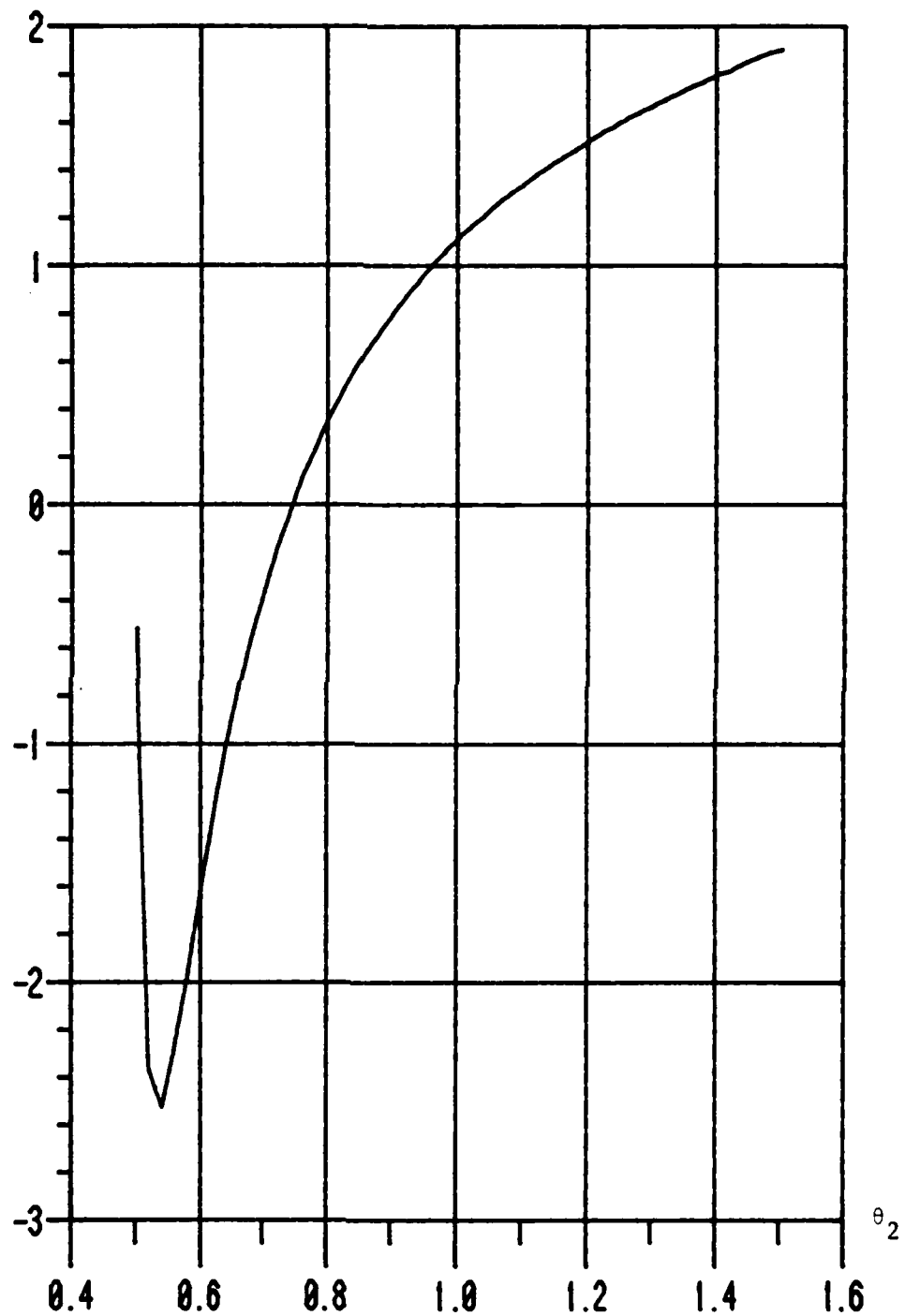
$J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$ 


Figure 14.  $J_{LS}(\text{1st order}) - J_{LS}(\text{2nd order})$  vs.  $\theta_2$ ,  $\theta_1=1$ ,  $a_2 \sim U[0.5, 1.5]$ .

of the real parameter from the nominal value increased. Disturbance attenuation was seen to be sacrificed as less parameter sensitivity was desired. Also, the controllers were biased towards those values of  $\theta$  associated with higher costs. Finally, it was seen that second-order controllers did not, in general, demonstrate much improvement over the first-order controllers.

# CHAPTER 4

## A MINIMAX SOLUTION

In this section, we seek to find the optimal controller for the value of the unknown parameter that corresponds to the worst-case model.

This problem is characterized by (8):

$$J(u_o, \theta_o) = \min_{u_t \in \Pi} \max_{\theta \in \Theta} J(u_t, \theta) \quad ,$$

where  $J(u_t, \theta)$  is given by (5). A way to calculate the solution is to find the optimal regulator for all possible models and then to maximize the corresponding cost over the parameter set. This problem is characterized by (9):

$$J(u_o, \theta_o) = \max_{\theta \in \Theta} \min_{u_t \in \Pi} J(u_t, \theta) \quad .$$

For the solutions of these two problems to be equivalent, a saddle point must exist at the solution point. Looze, Poor, et al. [7] have shown that a saddle point exists when the uncertainties are in the second-order statistics of the noise. However, some additional convexity assumptions are needed for the parameter set,  $\Theta$ , in this proof. We have made an attempt to use the same method to show the existence of a saddle point under the current assumptions. However, this cannot be done since, as seen later in the case of scalar systems, the saddle point does not exist for all types of parameter sets under consideration. Therefore, we state what conditions must be satisfied in Theorem 2, and then for scalar systems, we find the largest sets such that the saddle-point condition holds.

The examples from the previous section are then considered. The minimax solutions for these examples are calculated, and using the results for scalar systems, the minimax solution is shown to enjoy the saddle-point property for the sets under consideration. The performance of the minimax controller is compared to the performance of the least sensitive controller. It is generally seen that the least sensitive controllers exhibit superior performance over a wide range of parameter values in the set.

Finally, we include a note on the notation used here. A subscript " $\theta$ " denotes that the matrix in question is a function of the unknown parameter vector. A subscript of "o", or no subscript, indicates that the matrix is evaluated at the maximin solution.

#### 4.1. The Maximin Solution

To solve (9), the optimal linear regulator is determined for all  $\theta \in \Theta$ , and the resulting cost is maximized over  $\theta \in \Theta$ . The optimal linear regulator is [3]:

$$u_{\theta} = -W_u^{-1} G_{\theta}' P_{\theta} \hat{x}_t \quad (47a)$$

$$d\hat{x}_t = (F_{\theta} - G_{\theta}' W_u^{-1} G_{\theta} P_{\theta}) \hat{x}_t dt + \Sigma_{\theta} H_{\theta}' (LRL')^{-1} (dy_t - H_{\theta} \hat{x}_t dt) \quad (47b)$$

$$F_{\theta}' P_{\theta} + P_{\theta} F_{\theta} - P_{\theta} G_{\theta} W_u^{-1} G_{\theta}' P_{\theta} + W_x = 0 \quad (47c)$$

$$F_{\theta} \Sigma_{\theta} + \Sigma_{\theta} F_{\theta}' - \Sigma_{\theta} H_{\theta}' (LRL')^{-1} H_{\theta} \Sigma_{\theta} + KQK' = 0 \quad (47d)$$

The optimal cost is

$$J(u_{\theta}, \theta) = \text{tr}[P_{\theta} KQK' + P_{\theta} G_{\theta} W_u^{-1} G_{\theta}' P_{\theta} \Sigma_{\theta}] \quad (48)$$

Maximizing (48) over  $\theta \in \Theta$  with  $P_\theta$  and  $\Sigma_\theta$  given by (47c) and (47d) gives the maximin point,  $(u_0, \theta_0)$ . The cost at this point is

$$J(u_0, \theta_0) = \text{tr}[P_0 K Q K' + P_0 G_0 W_u^{-1} G_0' P_0 \Sigma_0] \quad (49)$$

In order to investigate the existence of a saddle point at this solution, an expression for the cost of the maximin control at other values of  $\theta$  is needed. By applying the control, (47) evaluated at  $\theta_0$ , to the system, (3), the following closed-loop model is formed. Let

$$\hat{S} = \begin{pmatrix} F_\theta & -G_\theta W_u^{-1} G_\theta' P \\ \Sigma H' (L R L')^{-1} H_\theta & F - G W_u^{-1} G' P - \Sigma H' (L R L')^{-1} H \end{pmatrix} \quad (50a)$$

$$\hat{N} = \begin{pmatrix} K & 0 \\ 0 & -\Sigma H' (L R L')^{-1} L \end{pmatrix} \quad (50b)$$

$$\hat{W} = \begin{pmatrix} W_x & 0 \\ 0 & P G W_u^{-1} G' P \end{pmatrix} \quad (50c)$$

Then the closed-loop model is

$$\begin{pmatrix} dx_t \\ d\hat{x}_t \end{pmatrix} = \hat{S} \begin{pmatrix} x_t \\ \hat{x}_t \end{pmatrix} dt + \hat{N} \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} \quad (51)$$

The corresponding cost is (under the assumption that  $\hat{S}$  is a stable matrix for all  $\theta \in \Theta$  — which will be justified later)

$$J(u_0, \theta) = \text{tr}(\hat{W} \hat{X}) \quad (52)$$

where  $\hat{X}$  is the positive definite solution to

$$\hat{S} \hat{X} + \hat{X} \hat{S}' + \hat{N} \hat{V} \hat{N}' = 0 \quad (53)$$

Let

$$\hat{X} = \begin{pmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_2' & \hat{X}_3 \end{pmatrix}.$$

Then Equation (52) reduces to

$$J(u_0, \theta) = \text{tr}[W_X \hat{X}_1 + PGW_u^{-1} G' P \hat{X}_3] \quad (54)$$

Also, (53) leads to three equations:

$$F_\theta \hat{X}_1 + \hat{X}_1 F_\theta' - G_\theta W_u^{-1} G' P \hat{X}_2' - \hat{X}_2 P G W_u^{-1} G_\theta' + K Q K' = 0 \quad (55a)$$

$$F_\theta \hat{X}_2 - G_\theta W_u^{-1} G' P \hat{X}_3 + \hat{X}_1 H_\theta' (LRL')^{-1} H \Sigma + \hat{X}_2 (F - G W_u^{-1} G' P - \Sigma H' (LRL')^{-1} H)' = 0 \quad (55b)$$

$$(F - G W_u^{-1} G' P - \Sigma H' (LRL')^{-1} H) \hat{X}_3 + \hat{X}_3 (F - G W_u^{-1} G' P - \Sigma H' (LRL')^{-1} H)' + \Sigma H' (LRL')^{-1} H \Sigma + \Sigma H' (LRL')^{-1} H_\theta \hat{X}_2 + \hat{X}_2 H_\theta' (LRL')^{-1} H \Sigma = 0 \quad (55c)$$

Equations (54) and (55) represent the cost incurred by the maximin controller for models that correspond to all the different parameter values in the set.

#### 4.2. The Minimax Solution - Existence of a Saddle Point

The solution to the minimax problem, (8), exists, and is equal to (9) if a saddle point exists at the solution,  $J(u_0, \theta_0)$ . The saddle point is characterized in the following theorem.

**Theorem 1 [2]:** There exists a pair  $(u_0, \theta_0) \in \Pi \times \Theta$  satisfying the saddle-point condition:

$$J(u_0, \theta) \leq J(u_0, \theta_0) \leq J(u_t, \theta_0) \quad \forall u_t \in \Pi, \theta \in \Theta, \quad (56)$$

if and only if the values of (8) and (9) are equal.

It is necessary to show that condition (56) holds. The right-hand side follows immediately from the fact that  $u_o$  is the optimal linear regulator for  $\theta_o$ . Looze, Poor, et al. [7] prove the left-hand side for their case by establishing the existence of the maximin solution to (9), and then showing that the condition that the Fréchet differential of  $J(u_t, \theta)$  at  $(u_o, \theta_o)$  is nonpositive for all  $\theta \in \Theta$  is equivalent to the left-hand side of (56).

To follow similar logic in the present case, a further restriction must be made on the parameter set. Specifically, we assume that  $\theta$  may be split into three vectors,  $\theta_F$ ,  $\theta_G$  and  $\theta_H$ , where the following conditions hold:

$$\begin{aligned} F_\theta &= F(\theta_F) & \theta_F &\in \Theta_F \\ G_\theta &= G(\theta_G) & \theta_G &\in \Theta_G \\ H_\theta &= H(\theta_H) & \theta_H &\in \Theta_H \end{aligned} \quad (57)$$

and  $\Theta_F$ ,  $\Theta_G$  and  $\Theta_H$  are disjoint, convex sets, and  $\Theta_F \times \Theta_G \times \Theta_H = \Theta$ .

Since the pairs  $(F_\theta, G_\theta)$  and  $(F_\theta, H_\theta)$  are stabilizable and detectable, respectively, and since  $u_o$  is an optimal regulator,  $J(u_o, \theta)$  is bounded and continuous in  $\theta$ . Therefore, since  $\Theta$  is compact, the maximin solution to (9) exists.

We now make the dependence of  $J(u_o, \theta)$  more explicit:

$$J(u_o, \theta) = J(u_o, F_\theta, G_\theta, H_\theta) \quad , \quad (58)$$

and we will refer to  $(F_\theta, G_\theta, H_\theta)$  as the point corresponding to  $(\theta)$ . Let  $(u_o, \theta_o) = (u_o, F, G, H)$  be the maximin solution. Then the Fréchet differential of (48) with respect to  $F_\theta, G_\theta, H_\theta$  at  $(F, G, H)$  must be nonpositive in every direction into  $\Theta$ . The Fréchet differential of (48) is now calculated with respect to each of the parameter vectors.

We start with Equation (48),

$$J(u_0, F_\theta, G_\theta, H_\theta) = \text{tr}[P_\theta K Q K' + P_\theta G_\theta W_u^{-1} G_\theta' P_\theta \Sigma_\theta]$$

The differential of  $J(u_0, F_\theta, G_\theta, H_\theta)$  with respect to  $F_\theta$  at  $(F, G, H)$  is

$$\delta J(u_0, F, G, H; \Delta F) = \text{tr}[\delta P K Q K' + \delta P G W_u^{-1} G' P \Sigma + P G W_u^{-1} G' \delta P \Sigma + P G W_u^{-1} G' P \delta \Sigma] \quad (59)$$

where  $\delta P$  and  $\delta \Sigma$  are the Fréchet differentials of  $P_\theta$  and  $\Sigma_\theta$  at  $(F, G, H)$ , and can be calculated using (47c) and (47d).

For  $\delta P$  we use (47c):

$$\begin{aligned} F'P + P'F - P'G W_u^{-1} G'P + W_x &= 0 \\ \Delta F'P + F'\delta P + \delta P'F + P\Delta F - \delta P G W_u^{-1} G'P - P G W_u^{-1} G'\delta P &= 0 \\ (F - G W_u^{-1} G'P)' \delta P + \delta P (F - G W_u^{-1} G'P) + \Delta F'P + P\Delta F &= 0 \end{aligned}$$

Let  $\hat{A} = F - G W_u^{-1} G'P$ . Then,

$$\delta P = \int_0^\infty e^{\hat{A}'t} (\Delta F'P + P\Delta F) e^{\hat{A}t} dt \quad (60)$$

For  $\delta \Sigma$  we use (47d):

$$\begin{aligned} F\Sigma + \Sigma F' - \Sigma H'(LRL')^{-1}H\Sigma + KQK' &= 0 \\ \Delta F\Sigma + F\delta\Sigma + \delta\Sigma F' + \Delta F\Sigma - \delta\Sigma H'(LRL')^{-1}H\Sigma - \Sigma H'(LRL')^{-1}H\delta\Sigma &= 0 \\ (F - \Sigma H'(LRL')^{-1}H)\delta\Sigma + \delta\Sigma (F - \Sigma H'(LRL')^{-1}H)' + \Delta F\Sigma + \Sigma\Delta F' &= 0 \end{aligned}$$

Let  $\hat{B} = F - \Sigma H'(LRL')^{-1}H$ . Then,

$$\delta\Sigma = \int_0^\infty e^{\hat{B}t} (\Delta F\Sigma + \Sigma\Delta F') e^{\hat{B}'t} dt \quad (61)$$



Substituting (61) and (60) into (59),

$$\begin{aligned}
 \delta J(u_0, F, G, H ; \Delta F) &= \text{tr}\{\delta P[KQK' + G W_u^{-1} G' P \Sigma + \Sigma P G W_u^{-1} G'] + [P G W_u^{-1} G' P] \delta \Sigma\} \\
 &= \text{tr}\left\{\int_0^\infty e^{\hat{A}t} [\Delta F' P + P \Delta F] e^{\hat{A}t} dt [KQK' + G W_u^{-1} G' P \Sigma + \Sigma P G W_u^{-1} G'] \right. \\
 &\quad \left. + [P G W_u^{-1} G' P] \int_0^\infty e^{\hat{B}t} [\Delta F \Sigma + \Sigma \Delta F'] e^{\hat{B}t} dt\right\} \\
 &= \text{tr}\left\{[\Delta F' P + P \Delta F] \int_0^\infty e^{\hat{A}t} [KQK' + G W_u^{-1} G' P \Sigma + \Sigma P G W_u^{-1} G'] e^{\hat{A}t} dt \right. \\
 &\quad \left. + [\Delta F \Sigma + \Sigma \Delta F'] \int_0^\infty e^{\hat{B}t} [P G W_u^{-1} G' P] e^{\hat{B}t} dt\right\}
 \end{aligned}$$

$$\delta J(u_0, F, G, H ; \Delta F) = 2 \text{tr}\{[M_{F1} P + \Sigma M_{F2}] \Delta F\} \quad (62)$$

$$(F - G W_u^{-1} G' P) M_{F1} + M_{F1} (F - G W_u^{-1} G' P)' + KQK' + G W_u^{-1} G' P \Sigma + \Sigma P G W_u^{-1} G' = 0 \quad (63a)$$

$$(F - \Sigma H' (LRL')^{-1} H)' M_{F2} + M_{F2} (F - \Sigma H' (LRL')^{-1} H) + P G W_u^{-1} G' P = 0 \quad (63b)$$

Equation (62) with Equations (63) describes the Fréchet differential of  $J(u_0, F_\theta, G_\theta, H_\theta)$  with respect to  $F_\theta$  at  $(F, G, H)$ .

In order to find the differential with respect to  $G_\theta$  at  $(F, G, H)$  we again start with (48):

$$J(u_0, F_\theta, G_\theta, H_\theta) = \text{tr}[P_\theta KQK' + P_\theta G_\theta W_u^{-1} G_\theta' P_\theta \Sigma_\theta]$$

Then, using (47c) and (47d),

$$J(u_0, F_\theta, G_\theta, H_\theta) = \text{tr}[W_x \Sigma_\theta + P_\theta \Sigma_\theta H_\theta' (LRL')^{-1} H_\theta \Sigma_\theta]$$

The differential of  $J(u_0, F_\theta, G_\theta, H_\theta)$  with respect to  $G_\theta$  at  $(F, G, H)$  is

$$\delta J(u_0, F, G, H ; \Delta G) = \text{tr}\{\delta P[\Sigma H' (LRL')^{-1} H \Sigma]\} \quad (64)$$

where  $\delta P$  is now the Fréchet differential of  $P_\theta$  with respect to  $G_\theta$  at  $(F, G, H)$ , and can be calculated using (47c):

$$F'\delta P + \delta P F - \delta P G W_u^{-1} G' P - P \Delta G W_u^{-1} G' P - P G W_u^{-1} \Delta G' P - P G W_u^{-1} G' \delta P = 0$$

$$(F - G W_u^{-1} G' P)'\delta P + \delta P (F - G W_u^{-1} G' P) - P \Delta G W_u^{-1} G' P - P G W_u^{-1} \Delta G' P = 0$$

$$\delta P = \int_0^\infty e^{\hat{A}'t} [-P \Delta G W_u^{-1} G' P - P G W_u^{-1} \Delta G' P] e^{\hat{A}t} dt \quad (65)$$

Substituting (65) into (64)

$$\begin{aligned} \delta J(u_0, F, G, H; \Delta G) &= \text{tr} \left\{ \int_0^\infty e^{\hat{A}'t} [-P \Delta G W_u^{-1} G' P - P G W_u^{-1} \Delta G' P] e^{\hat{A}t} dt [\Sigma H' (LRL')^{-1} H \Sigma] \right\} \\ &= \text{tr} \{ [P \Delta G W_u^{-1} G' P + P G W_u^{-1} \Delta G' P] \int_0^\infty e^{\hat{A}t} [-\Sigma H' (LRL')^{-1} H \Sigma] e^{\hat{A}'t} dt \} \\ \delta J(u_0, F, G, H; \Delta G) &= 2 \text{tr} \{ [W_u^{-1} G' P M_G] \Delta G \} \end{aligned} \quad (66)$$

$$(F - G W_u^{-1} G' P) M_G + M_G (F - G W_u^{-1} G' P)' - \Sigma H' (LRL')^{-1} H \Sigma = 0 \quad (67)$$

Equation (66) with (67) describes the Fréchet differential of  $J(u_0, F_\theta, G_\theta, H_\theta)$  with respect to  $G_\theta$  at  $(F, G, H)$ .

In order to find the differential of  $J(u_0, F_\theta, G_\theta, H_\theta)$  with respect to  $H_\theta$  at  $(F, G, H)$ , we again start with (48). Then,

$$\delta J(u_0, F, G, H; \Delta H) = \text{tr} \{ [P G W_u^{-1} G' P] \delta \Sigma \} \quad (68)$$

where  $\delta \Sigma$  is now the Fréchet differential of  $\Sigma_\theta$  with respect to  $H_\theta$  at  $(F, G, H)$ , and can be calculated using (47d):

$$\begin{aligned} F \delta \Sigma + \delta \Sigma F' - \delta \Sigma H' (LRL')^{-1} H \Sigma - \Sigma \Delta H' (LRL')^{-1} H \Sigma - \Sigma H' (LRL')^{-1} \Delta H \Sigma \\ - \Sigma H' (LRL')^{-1} H \delta \Sigma = 0 \end{aligned}$$

$$\begin{aligned} (F - \Sigma H' (LRL')^{-1} H) \delta \Sigma + \delta \Sigma (F - \Sigma H' (LRL')^{-1} H) - \Sigma \Delta H' (LRL')^{-1} H \Sigma \\ - \Sigma H' (LRL')^{-1} \Delta H \Sigma = 0 \end{aligned}$$

$$\delta \Sigma = \int_0^\infty e^{\hat{B}t} [-\Sigma \Delta H' (LRL')^{-1} H \Sigma - \Sigma H' (LRL')^{-1} \Delta H \Sigma] e^{\hat{B}'t} dt \quad (69)$$

Substituting (69) into (68),

$$\begin{aligned}\delta J(u_0, F, G, H; \Delta H) &= \text{tr}\{[PGW_u^{-1}G^*P] \int_0^\infty e^{\hat{B}t} [-\Sigma \Delta H^* (LRL^*)^{-1} H \Sigma - \Sigma H^* (LRL^*)^{-1} \Delta H \Sigma] \\ &\quad \cdot e^{\hat{B}^*t} dt\} \\ &= \text{tr}\{[\Sigma \Delta H^* (LRL^*)^{-1} H \Sigma + \Sigma H^* (LRL^*)^{-1} \Delta H \Sigma] \int_0^\infty e^{\hat{B}^*t} [-PGW_u^{-1}G^*P] e^{\hat{B}t} dt\} \\ \delta J(u_0, F, G, H; \Delta H) &= 2\text{tr}\{[\Sigma M_H \Sigma H^* (LRL^*)^{-1}] \Delta H\} \quad (70)\end{aligned}$$

$$(F - \Sigma H^* (LRL^*)^{-1} H) M_H + M_H (F - \Sigma H^* (LRL^*)^{-1} H) - PGW_u^{-1}G^*P = 0 \quad (71)$$

Equation (70) with (71) describes the Fréchet differential of  $J(u_0, F_\theta, G_\theta, H_\theta)$  with respect to  $H_\theta$  at  $(F, G, H)$ . This completes the calculation of the Fréchet differentials.

Consider an arbitrary point,  $(F_\theta, G_\theta, H_\theta)$ , in  $\Theta$ . Since  $\Theta_F$ ,  $\Theta_G$  and  $\Theta_H$  are convex, the line segment joining  $(F, G, H)$  and  $(F_\theta, G_\theta, H_\theta)$  is in  $\Theta$  and

$$(\Delta F, \Delta G, \Delta H) = (F_\theta - F, G_\theta - G, H_\theta - H) \quad (72)$$

is a direction into  $\Theta$ . Since  $(F, G, H)$  is the maximin solution, differentials (62), (66) and (70) are nonpositive. If by setting differentials (62), (66) and (70) nonpositive we can show that the left-hand side of (56) is true, then a saddle point exists. This can be summarized in the following theorem.

**Theorem 2:** Let the Fréchet differentials (62), (66), (70) corresponding to the maximin solution be nonpositive:

$$\text{tr}\{[M_{F1}P + \Sigma M_{F2}](F_\theta - F)\} \leq 0 \quad \forall F_\theta \in \Theta_F \quad (73a)$$

$$\text{tr}\{[W_u^{-1}G^*PM_GP](G_\theta - G)\} \leq 0 \quad \forall G_\theta \in \Theta_G \quad (73b)$$

$$\text{tr}\{[\Sigma M_H \Sigma H^* (LRL^*)^{-1}](H_\theta - H)\} \leq 0 \quad \forall H_\theta \in \Theta_H \quad (73c)$$

where  $M_{F1}$ ,  $M_{F2}$ ,  $M_G$ ,  $M_H$  are given by (63a), (63b) (67), (71).

Consider the left-hand side of (56) in view of (54).

$$J(u_0, F_\theta, G_\theta, H_\theta) \leq J(u_0, F, G, H) \quad (74a)$$

$$\text{tr}[W_x \hat{X}_{1\theta} + PGW_u^{-1} G^T P \hat{X}_{3\theta}] \leq \text{tr}[W_x \hat{X}_{10} + PGW_u^{-1} G^T P \hat{X}_{30}] \quad (74b)$$

where  $\hat{X}_{1\theta}$  and  $\hat{X}_{3\theta}$  are given by Equations (55), and  $\hat{X}_{10}$  and  $\hat{X}_{30}$  are given by Equations (55), evaluated at  $\theta_0$ .

If (73) implies (74), then a saddle point exists, and the solutions of (8) and (9) are equal.

We have shown that (73) is true since the maximin solution exists. Also, the right-hand side of (56) is known to be true. Therefore, if we can show that the left-hand side of (56) is true using (73), Theorem 1 can be used to establish the existence of a saddle point and the equivalence of the two solutions.

Looze, Poor, et al. [7] take advantage of the fact that the system parameters are known, and by use of the transformation,  $e_t = x_t - \hat{x}_t$ , an error model can be formed. Using this error model, an expression for  $J(u_0, \theta)$  can be found such that the corresponding equations to (73) and (74) are identical. However, under the present circumstances, this comparison is not possible. This is true because uncertainty in the system parameters makes the formation of an error model impossible. Therefore, Equation (74b) is dissimilar to Equations (73), and it is not immediately apparent that (73) implies (74). Indeed, as we see in the case of scalar systems, (73) does not imply (74) for all types of sets in the class under consideration. We now undertake the study of this problem for scalar systems, and characterize all the sets in the class under consideration for which the saddle point exists.

### 4.3. Scalar Systems

The existence of saddle-point solutions for scalar systems will be studied for a large class of convex sets. The simplicity of scalar systems allows for a more thorough investigation of the properties in Theorem 2, and, therefore, we gain better insight into the structural properties of the problem at hand. First, we will show how Equations (73) may be used to characterize the maximum solution for the class of convex sets that satisfy the stabilizability and detectability assumptions. Then we will show that Equation (74) can be verified only for certain sets within this class.

First, the class of convex sets will be described. For scalar systems, the parameter vector  $(F_\theta, G_\theta, H_\theta)$  can have at most three scalar parameters that we represent by  $(f_\theta, g_\theta, h_\theta)$ . Convex sets that satisfy the conditions, (57), consist of three sets of intervals on the real line. To satisfy the stabilizability and detectability assumptions we require that these intervals do not contain the point zero. To allow for the possibility that one or two parameters are known, we allow one or two of these intervals to be reduced to a single point. Therefore, the class of sets we consider is  $\Theta$  such that:

$$\chi = \{\Theta : \Theta = \Theta_F \cup \Theta_G \cup \Theta_H\} , \quad (75)$$

and  $\Theta_F, \Theta_G, \Theta_H$  are each one of the following:

$$\Theta_F = \{f_\theta \in [f_a, f_b] \quad f_a < f_b < 0\} \quad (76a)$$

$$\text{or} \quad \Theta_F = \{f_\theta \in [f_a, f_b] \quad 0 < f_a < f_b\} \quad (76b)$$

$$\text{or} \quad \Theta_F = \{f_\theta = f \quad f \in \mathbb{R}, \quad f \neq 0\} \quad (76c)$$

$$\Theta_G = \{g_\theta \in [g_a, g_b] \quad g_a < g_b < 0\} \quad (77a)$$

$$\text{or} \quad \Theta_G = \{g_\theta \in [g_a, g_b] \quad 0 < g_a < g_b\} \quad (77b)$$

$$\text{or} \quad \Theta_G = \{g_\theta = g \quad g \in \mathbb{R}, g \neq 0\} \quad (77c)$$

$$\Theta_H = \{h_\theta \in [h_a, h_b] \quad h_a < h_b < 0\} \quad (78a)$$

$$\text{or} \quad \Theta_H = \{h_\theta \in [h_a, h_b] \quad 0 < h_a < h_b\} \quad (78b)$$

$$\text{or} \quad \Theta_H = \{h_\theta = h \quad h \in \mathbb{R}, h \neq 0\} \quad (78c)$$

and we need not consider the trivial case  $\Theta = \{(f, g, h)\}$ . Note that the assumed independence of  $f_\theta, g_\theta$  and  $h_\theta$  allows us to maximize  $J(u_0, f_\theta, g_\theta, h_\theta)$  with respect to  $f_\theta, g_\theta$ , and  $h_\theta$  one at a time.

The maximization of (48) over each set in the class,  $\chi$ , can be performed by using its Fréchet differentials in the Condition (73) of Theorem 2. We will consider each condition in (73) one at a time. First, consider Condition (73a) for  $f_\theta \in \Theta_F$  and with  $\Theta_F$  as in (76a) or (76b) and noting that Case (76c) is trivial:

$$[m_{F1}^P + \Sigma m_{F2}](f_\theta - f) \leq 0 \quad \forall f_\theta \in \Theta_F \quad (79)$$

$$2(f - \frac{g_P^2}{w_u}) m_{F1} + k^2 q + 2 \frac{g_P^2 \Sigma}{w_u} = 0 \quad (80)$$

$$2(f - \frac{h^2 \Sigma}{2r}) m_{F2} + \frac{g_P^2}{w_u} = 0 \quad (81)$$

Equations (80) and (81) can be solved by direct division:

$$m_{F1} = \frac{-(k^2 q + \frac{2g_P^2 \Sigma}{w_u})}{2(f - \frac{g_P^2}{w_u})}$$

$$m_{F2} = \frac{-\frac{g_p^2}{w_u}}{2(f - \frac{h^2 \Sigma}{\ell^2 r})}$$

We know from the properties of optimal linear regulators that  $(f - \frac{g_p^2}{w_u})$  and  $(f - \frac{h^2 \Sigma}{\ell^2 r})$  are stable, which in this case means they are strictly negative. Therefore,  $m_{F1}$  and  $m_{F2}$  are positive, and  $m_{F1}^P + m_{F2}^\Sigma$  is positive. Then, condition (79) is equivalent to

$$f_\theta \leq f \quad \forall f_\theta \in \Theta_F$$

The only such point,  $f$ , in sets  $\Theta_F$  in (76a) and (76b) is at the upper boundary. That is,

$$f = f_b \text{ for } f_\theta \in [f_a, f_b] \quad f_a < f_b < 0 \quad (82)$$

$$f = f_b \text{ for } f_\theta \in [f_a, f_b] \quad 0 < f_a < f_b \quad (83)$$

Now consider Condition (73b) for  $g_\theta \in \Theta_G$  and with  $\Theta_G$  in (77a) and (77b):

$$\frac{g_p^2}{w_u} m_G (g_\theta - g) \leq 0 \quad \forall g_\theta \in \Theta_G \quad (84)$$

$$2(f - \frac{g_p^2}{w_u}) m_G - \frac{h^2 \Sigma^2}{\ell^2 r} = 0 \quad (85)$$

(85) can be solved by direct division:

$$m_G = \frac{\frac{h^2 \Sigma^2}{\ell^2 r}}{2(f - \frac{g_p^2}{w_u})}$$

Since  $(f - \frac{g^2 p^2}{w_u}) < 0$ ,  $m_G < 0$ . Now consider  $\Theta_G$  in (77a). In this case  $g < 0$ , and therefore  $\frac{g p^2}{w_u} m_G > 0$ . Then Condition (84) is equivalent to:

$$g_\theta \leq g \quad \forall g_\theta \in \Theta_G$$

Therefore,  $g = g_b$ . With  $\Theta_G$  in (77b),  $g > 0$ , and therefore  $\frac{g^2 p^2}{w_u} m_G < 0$ . Then Condition (84) is equivalent to:

$$g_\theta \geq g \quad \forall g_\theta \in \Theta_G$$

Therefore,  $g$  occurs at the lower boundary,  $g_a$ . Summarizing the results:

$$g = g_b \text{ for } g_\theta \in [g_a, g_b] \quad g_a < g_b < 0 \quad (86)$$

$$g = g_a \text{ for } g_\theta \in [g_a, g_b] \quad 0 < g_a < g_b \quad (87)$$

Now, consider Condition (73c) for  $h_\theta \in \Theta_H$  and with  $\Theta_H$  in (78a) and (78b):

$$\frac{h \Sigma^2}{\ell^2 r} m_H (h_\theta - h) \leq 0 \quad \forall h_\theta \in \Theta_H \quad (88)$$

$$2(f - \frac{h^2 \Sigma}{\ell^2 r}) m_H - \frac{g^2 p^2}{w_u} = 0 \quad (89)$$

(89) can be solved by direct division:

$$m_H = \frac{\frac{g^2 p^2}{w_u}}{2(f - \frac{h^2 \Sigma}{\ell^2 r})}$$



Since  $(f - \frac{h^2 \Sigma}{\ell^2 r}) < 0$ ,  $m_H < 0$ . Now consider  $\Theta_H$  in (78a). In this case  $h < 0$ , and therefore  $\frac{h \Sigma^2}{\ell^2 r} m_H > 0$ . Then Condition (88) is equivalent to:

$$h_\theta \leq h \quad \forall h_\theta \in \Theta_H.$$

Therefore,  $h$  occurs at the upper boundary,  $h_b$ . With  $\Theta_H$  in (78b),  $h > 0$ , and therefore  $\frac{h \Sigma^2}{\ell^2 r} m_H < 0$ . Then Condition (88) is equivalent to:

$$h_\theta \geq h \quad \forall h_\theta \in \Theta_H.$$

Therefore  $h$  occurs at the lower boundary,  $h_a$ . Summarizing the results:

$$h = h_b \text{ for } h_\theta \in [h_a, h_b] \quad h_a < h_b < 0 \quad (90)$$

$$h = h_a \text{ for } h_\theta \in [h_a, h_b] \quad 0 < h_a < h_b \quad (91)$$

(82) and (83), (86) and (87), (90) and (91) are the solutions for sets (76), (77) and (78), respectively. The solution for sets in class  $\chi$  can be found by applying solutions (82) and (83), (86) and (87), (90) and (91) one at a time whenever a parameter is unknown. This completes the solution of the maximin problem for sets in the class,  $\chi$ .

We now wish to verify the saddle-point condition in Theorem 2 by showing the relation, (74), holds. It will be seen that saddle points do not exist for all sets in the class,  $\chi$ . However, we can find a class of sets within  $\chi$  for which (74) can be verified. Recall that the relation, (74), is part of the saddle-point condition, (56), in view of the expression for  $J(u_0, \theta)$  in (54). Therefore, to verify (74) we need an explicit expression for  $J(u_0, \theta) = J(u_0, f_\theta, g_\theta, h_\theta)$ . Towards this goal we start with (54),

$$J(u_0, f_\theta, g_\theta, h_\theta) = w_x \hat{X}_1 + \frac{g_p^2}{w_u} \hat{X}_3 \quad (92)$$

and  $\hat{X}_1$  and  $\hat{X}_3$  are solutions of Equations (55):

$$2f_\theta \hat{X}_1 - \frac{2g_\theta g_p \hat{X}_2}{w_u} + k^2 q = 0 \quad (93)$$

$$(f_\theta + f - \frac{g_p^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) \hat{X}_2 - \frac{g_\theta g_p \hat{X}_3}{w_u} + \frac{h_\theta h \Sigma \hat{X}_1}{\ell^2 r} = 0 \quad (94)$$

$$2(f - \frac{g_p^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) \hat{X}_3 + \frac{2h_\theta h \Sigma \hat{X}_2}{\ell^2 r} + \frac{h^2 \Sigma^2}{\ell^2 r} = 0 \quad (95)$$

Using (93),

$$\hat{X}_1 = \frac{-k^2 q}{2f_\theta} + \frac{g_\theta g_p \hat{X}_2}{f_\theta w_u} \quad (96)$$

Using (95),

$$\hat{X}_3 = \frac{\frac{-h^2 \Sigma^2}{\ell^2 r}}{2(f - \frac{g_p^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})} + \frac{\frac{-h_\theta h \Sigma}{\ell^2 r} \hat{X}_2}{(f - \frac{g_p^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})} \quad (97)$$

Substituting into (94),  $\hat{X}_2$  may be found.

$$\left[ f_\theta + f - \frac{g_p^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} + \frac{\frac{g_\theta g_p h_\theta h \Sigma}{w_u \ell^2 r}}{(f - \frac{g_p^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})} + \frac{g_\theta g_p h_\theta h \Sigma}{f_\theta w_u \ell^2 r} \right] \hat{X}_2$$

$$+ \frac{g_\theta g_p h^2 \Sigma^2}{2w_u \ell^2 r (f - \frac{g_p^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})} - \frac{h_\theta h \Sigma k^2 q}{f_\theta 2 \ell^2 r} = 0$$

Solving, and after some rearrangement,

$$\hat{x}_2 = \frac{\frac{-f_\theta g_\theta g_P h^2 \Sigma^2}{2w_u \ell^2 r} + \frac{h_\theta h \Sigma k^2 q (f_\theta - \frac{g_P^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})}{2\ell^2 r}}{(f_\theta + f - \frac{g_P^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) [f_\theta (f - \frac{g_P^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) + \frac{g_\theta g_P h_\theta h \Sigma}{w_u \ell^2 r}]} \quad (98)$$

We substitute (98) into (96) and (97), and then into (92) to find an expression for  $J(u_\theta, f_\theta, g_\theta, h_\theta)$ . The final result is

$$J(u_\theta, f_\theta, g_\theta, h_\theta) = \frac{-1}{[f_\theta + f - \frac{g_P^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}] [f_\theta (f - \frac{g_P^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) + \frac{g_\theta g_P h_\theta h \Sigma}{w_u \ell^2 r}]} \cdot \left[ \frac{h_\theta^2 h^2 \Sigma^2 g_P^2 k^2 q}{2\ell^4 r^2 w_u} + \frac{g_\theta^2 g_P^2 h^2 \Sigma^2 w_x}{2w_u \ell^2 r} + \frac{g_\theta g_P h_\theta h \Sigma}{2w_u \ell^2 r} \left( w_x k^2 q + \frac{g_P^2 h^2 \Sigma^2}{w_u \ell^2 r} \right) \right. \\ \left. + \frac{f_\theta^2 g_P^2 h^2 \Sigma^2}{2w_u \ell^2 r} + \frac{f_\theta}{2} \left( w_x k^2 q + \frac{g_P^2 h^2 \Sigma^2}{w_u \ell^2 r} \right) \left( f - \frac{g_P^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} \right) + \frac{w_x k^2 q}{2} \left( f - \frac{g_P^2}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} \right)^2 \right] \quad (99)$$

The properties of Equation (99) may be used to investigate the saddle-point properties of the maximin solutions. We can show that (99) is decreasing away from the maximin solution locally, so that there always exist some sets in  $X$  such that (74) is verified. We can also show that as the sets get larger the saddle-point condition breaks down in some cases, since, as we move away from the maximin point, (99) may start to increase again until its value becomes greater than the maximin value.

To show that (99) is decreasing away from the maximin solution, we first note that since  $J(u_o, \theta)$  in (48) is the optimal cost at any value of  $\theta$ , (99) is bounded below by (48) at any value of  $f_\theta, g_\theta$  and  $h_\theta$ . From this fact, and the fact that at the maximin point (99) and (48) are equal and continuous, the derivatives of (99) are equal to the derivatives of (48) at the maximin point. The derivative relations (79), (84) and (88) in view of (80), (81), (85) and (89) indicate that (99) is decreasing as we move away from the maximin point into a set in  $\chi$ . Therefore, there exists a set belonging to  $\chi$  in which (99) is decreasing in all directions from the maximin point into the set. Since (99) is continuous in this set, (74) is satisfied, and therefore this is a set in  $\chi$  such that a saddle point exists.

We have shown that there exist sets in  $\chi$  such that the maximin solution satisfies the saddle-point condition. In the following, we show that for a given maximin solution, a largest set in the class,  $\chi$ , can be found such that the saddle-point condition holds. This set is found by examining the topology of the level surface,

$$J(u_o, f_\theta, g_\theta, h_\theta) = J(u_o, f, g, h) \quad (100)$$

We show that this equation describes the boundary of the largest convex compact set such that the saddle-point condition holds. However, this set does not belong to the class,  $\chi$ , under consideration. Therefore, we find the largest set in  $\chi$  that is contained in this largest convex set. We find that there are never any limits to the size of  $\Theta_F$ , but there are always limits on the size of  $\Theta_G$  and  $\Theta_H$ .

We start with Equation (100) with  $J(u_o, f, g, h)$  given by (49):

$$\begin{aligned}
 J(u_o, f_\theta, g_\theta, h_\theta) &= (k^2_{qP} + \frac{g^2_P h^2_\Sigma}{w_u}) \\
 &= \frac{(k^2_{qP} + \frac{g^2_P h^2_\Sigma}{w_u}) (f_\theta^2 A + \frac{f_\theta g_\theta g P h_\theta h \Sigma}{w_u \ell^2_r} + f_\theta A^2 + \frac{g_\theta g P h_\theta h \Sigma A^2}{w_u \ell^2_r})}{(f_\theta + A) (f_\theta A + \frac{g_\theta g P h_\theta h \Sigma}{w_u \ell^2_r})}
 \end{aligned}$$

where

$$A = f - \frac{g^2_P}{w_u} - \frac{h^2_\Sigma}{\ell^2_r} \quad (101)$$

Using (99), and cancelling out the common denominator on both sides we obtain the result:

$$K_1 f_\theta^2 + K_2 f_\theta + K_3 f_\theta g_\theta h_\theta + K_4 g_\theta h_\theta + K_5 h_\theta^2 + K_6 g_\theta^2 + K_7 = 0 \quad (102)$$

with

$$K_1 = \frac{g^2_P h^2_\Sigma}{2 w_u \ell^2_r} + (k^2_{qP} + \frac{g^2_P h^2_\Sigma}{w_u}) (f - \frac{g^2_P}{w_u} - \frac{h^2_\Sigma}{\ell^2_r}) \quad (103a)$$

$$K_2 = \frac{1}{2} (w_x k^2_{qP} + \frac{g^2_P h^2_\Sigma}{w_u \ell^2_r}) (f - \frac{g^2_P}{w_u} - \frac{h^2_\Sigma}{\ell^2_r}) + (k^2_{qP} + \frac{g^2_P h^2_\Sigma}{w_u}) (f - \frac{g^2_P}{w_u} - \frac{h^2_\Sigma}{\ell^2_r})^2 \quad (103b)$$

$$K_3 = \frac{g P h \Sigma}{w_u \ell^2_r} (k^2_{qP} + \frac{g^2_P h^2_\Sigma}{w_u}) \quad (103c)$$

$$K_4 = \frac{g P h \Sigma}{w_u \ell^2_r} \left[ \frac{1}{2} (w_x k^2_{qP} + \frac{g^2_P h^2_\Sigma}{w_u \ell^2_r}) + (k^2_{qP} + \frac{g^2_P h^2_\Sigma}{w_u}) (f - \frac{g^2_P}{w_u} - \frac{h^2_\Sigma}{\ell^2_r}) \right] \quad (103d)$$

$$K_5 = \frac{g_P^2 h_\Sigma^2 k_q^2}{2w_u^4 r^2} \quad (103e)$$

$$K_6 = \frac{g_P^2 h_\Sigma^2 w_x^2}{2w_u^2 \ell^2 r} \quad (103f)$$

$$K_7 = \frac{w_x k_q^2}{2} \left( f - \frac{g_P^2}{w_u} - \frac{h_\Sigma^2}{\ell^2 r} \right)^2 \quad (103g)$$

Equation (102) with (103) describes a surface in  $\mathbb{R}^3$ , part of which describes the boundary of the largest convex set such that the saddle-point relation, (74), holds. This surface, of course, contains the point  $(f, g, h)$  and turns out to be hyperboloid in shape, although Equation (102) is not quite quadratic. To describe this surface precisely, we will investigate the contours of (102) in the  $f_\theta$ - $g_\theta$  plane, the  $f_\theta$ - $h_\theta$  plane, and the  $g_\theta$ - $h_\theta$  plane at values  $h_\theta = h$ ,  $g_\theta = g$  and  $f_\theta = f$ , respectively.

We start by investigating the contour in the  $f_\theta$ - $g_\theta$  plane with  $h_\theta = h$ . This situation is equivalent to taking  $\Theta_H$  as in (78c). Equation (102) then reduces to a quadratic equation of the form:

$$K_1 f_\theta^2 + K_2 f_\theta + K_3 h f_\theta g_\theta + K_4 h g_\theta + K_6 g_\theta^2 + K_5 h_\theta^2 + K_7 = 0 \quad (104)$$

A typical graph of Equation (104) is shown in Figure 15. This graph exhibits three basic characteristics we need in order to construct the largest set in  $\chi$  that enjoys the saddle-point property. These characteristics are:

(Ia) The graph of (104) is a hyperbola.

(Ib) Curve A intersects the line  $f_\theta = f$  at exactly one other point,  $g_\theta = g_{\max}$ , where  $g_{\max}$  is always in a set  $\Theta_G$  in  $\chi$ .

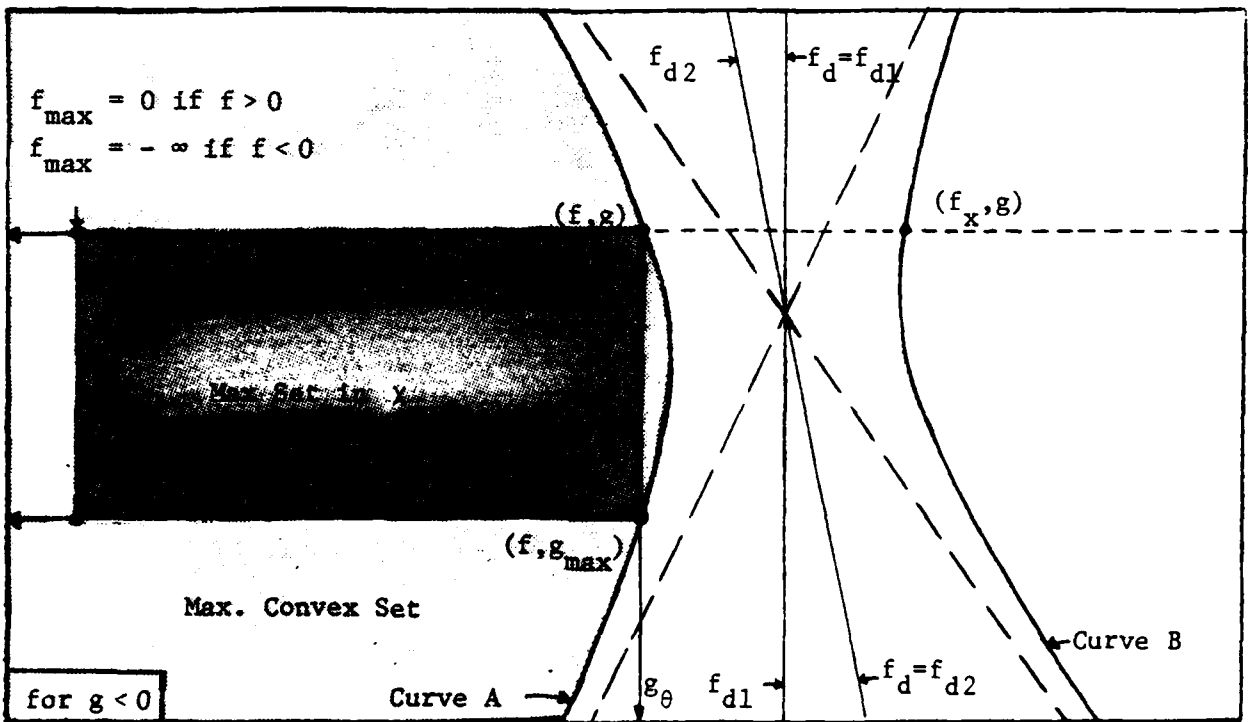


Figure 15. Contours of  $J(f_\theta, g_\theta, h) = J(f, g, h)$  in the  $f_\theta - g_\theta$  plane.

(Ic) Curve A never intersects the line,  $g_\theta = g$ , at a point other than the maximin point,  $(f, g, h)$ .

The shaded area to the left of Curve A is a convex set in the  $f_\theta - g_\theta$  plane, and is the largest set that enjoys the saddle-point property, since we have already shown that  $J(u_0, f_\theta, g_\theta, h_\theta)$  is decreasing at  $(f, g, h)$  in all directions into the area with darker shading. However, this set is not rectangular and therefore is not in  $\chi$ .

Characteristics (Ia), (Ib) and (Ic) enable us to construct the largest set in  $\chi$  that enjoys the saddle-point property. This set is shown in Figure 15 with darker shading.

It remains to show that Characteristics (Ia), (Ib) and (Ic) are true. Necessary and sufficient conditions for (Ia), (Ib) and (Ic) to be true are:

- (I1) The graph of Equation (104) is either a hyperbola or an ellipse.
- (I2) There exist two asymptotes of  $J(u_0, f_\theta, g_\theta, h)$  in the  $f_\theta - g_\theta$  plane. They are a pair of intersecting lines.
- (I3) There always exists exactly one other point besides  $(f, g, h)$  such that  $J(u_0, f, g_\theta, h) = J(u_0, f, g, h)$ .
- (I4) This point is always in some set  $\Theta_C$  of the class,  $\chi$ , under consideration, and is the point  $(f, g_{\max}, h)$  as represented in Figure 15.
- (I5) There always exists exactly one other point besides  $(f, g, h)$  such that  $J(u_0, f_\theta, g, h) = J(u_0, f, g, h)$ .
- (I6) This point is always on the opposite side of the asymptotes of  $J(u_0, f_\theta, g_\theta, h)$ , and is the point  $(f_x, g, h)$  on Curve B in Figure 15.



Conditions (I1), (I2), (I5) and (I6) verify Characteristic (Ia). Conditions (I3) and (I4) verify Characteristic (Ib), and Conditions (I2), (I5) and (I6) verify Characteristic (Ic). We now show that Conditions (I1) through (I6) are valid and discuss the precise reason why they prove that Characteristics (Ia), (Ib) and (Ic) are valid.

Since Equation (104) is quadratic in  $f_\theta$  and  $g_\theta$ , and since  $K_1 \neq 0$ ,  $K_5 \neq 0$ ,  $K_6 \neq 0$  and  $K_7 \neq 0$ , its graph is either a hyperbola or an ellipse. Therefore, Condition (I1) holds.

From Equation (99) we see that the denominator equals zero for certain values of  $f_\theta, g_\theta$  and  $h_\theta$ . Setting the first factor equal to zero we find that an asymptote occurs at

$$f_\theta + f - \frac{g_\theta^2 p}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} = 0 \quad (105)$$

The graph of (105) in the  $f_\theta$ - $g_\theta$  plane is a vertical line. Setting the second factor in (99) equal to zero we find that another asymptote occurs at

$$f_\theta \left( f - \frac{g_\theta^2 p}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} \right) + \frac{g_\theta g_\theta^p h_\theta h \Sigma}{w_u \ell^2 r} = 0 \quad (106)$$

The graph of (106) with  $h_\theta = h$  in the  $f_\theta$ - $g_\theta$  plane is a line. Since the slope of (106) is different than that for (105), these lines intersect at some point. Since the numerator of Equation (99) is nonzero for values of  $f_\theta$  and  $g_\theta$  satisfying (105) and (106), Condition (I2) holds.

Equation (104) evaluated at  $f_\theta = f$  is quadratic in  $g_\theta$ , and therefore has two solutions. One solution is  $g_\theta = g$ . Since this solution is real, and the coefficients of (104) are real, the other solution is also real, and Condition (I3) holds.

Equation (99) evaluated at  $f_\theta = f$  and  $h_\theta = h$  is of the form:

$$J(u_o, f, g_\theta, h) = \frac{G_1 + G_2 g_\theta + G_3 g_\theta^2}{(f + A)(fA + G_4 g_\theta)} \quad (107)$$

where

$$G_1 = -\left\{ \frac{h^4 \Sigma^2 g_P^2 k_q^2}{2\ell^4 r^2 w_u} + \frac{f^2 g_P^2 h^2 \Sigma^2}{2w_u \ell^2 r} + \frac{f}{2} (w_x k_q^2 + \frac{g_P^2 h^2 \Sigma^2}{w_u \ell^2 r}) (f - \frac{g_P^2}{w_u} - \frac{h^2 \Sigma^2}{\ell^2 r}) + \frac{w_x k_q^2}{2} (f - \frac{g_P^2}{w_u} - \frac{h^2 \Sigma^2}{\ell^2 r})^2 \right\} \quad (108a)$$

$$G_2 = -\left\{ \frac{g_P h^2 \Sigma^2}{2w_u \ell^2 r} (w_x k_q^2 + \frac{g_P^2 h^2 \Sigma^2}{w_u \ell^2 r}) \right\} \quad (108b)$$

$$G_3 = -\left\{ \frac{g_P^2 h^2 \Sigma^2 w_x}{2w_u \ell^2 r} \right\} \quad (108c)$$

$$G_4 = \frac{g_P h^2 \Sigma^2}{2w_u \ell^2 r} \quad (108d)$$

We now show that Condition (I4) holds. We consider the cases,  $g < 0$  corresponding to  $\Theta_G$  in (77a) and  $g > 0$  corresponding to  $\Theta_G$  in (77b), separately.

Case (1):  $g < 0$  ,  $\Theta_G = \{g_\theta \in [g_a, g_b] \quad g_a < g_b < 0\}$

We have shown that the maximin point  $g$  equals  $g_b$  in this case. Therefore, to prove the validity of Condition (I4) in this case, it is necessary to show that there exists a  $g_{\max} \in (-\infty, g)$  such that  $J(u_o, f, g_{\max}, h) = J(u_o, f, g, h)$ . First, note that from Equation (84) and the subsequent discussion we have shown

that  $J(u_0, f, g_\theta, h)$  is increasing at  $(f, g, h)$ . Now consider Equation (106) evaluated at  $f_\theta = f$  and  $h_\theta = h$ , which yields a value of  $g_\theta$  which we denote by  $g_d$ :

$$g_d = \frac{-f(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})}{\frac{gph^2 \Sigma}{w_u \ell^2 r}} \quad (109)$$

$(f, g_d, h)$  is the only point where  $J(u_0, f, g_\theta, h)$  is discontinuous in  $g_\theta$ . We show that  $g < g_d$  so that  $J(u_0, f, g_\theta, h)$  is continuous over  $(-\infty, g)$ . Note that

$$g - g_d = \frac{\frac{g^2 Ph^2 \Sigma}{w_u \ell^2 r} + f(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})}{\frac{gph^2 \Sigma}{w_u \ell^2 r}} \quad (110)$$

Recall from Equations (47c) and (47d) that

$$\frac{g^2 P}{w_u} = \frac{2fP + w_x}{P} \quad (111)$$

$$\frac{h^2 \Sigma}{\ell^2 r} = \frac{2f\Sigma + k^2 q}{\Sigma} \quad (112)$$

Using (111) and (112) in the numerator of (110),

$$\begin{aligned} \frac{g^2 Ph^2 \Sigma}{w_u \ell^2 r} + f(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) &= \frac{g^2 Ph^2 \Sigma}{2w_u \ell^2 r} + \frac{g^2 Ph^2 \Sigma}{2w_u \ell^2 r} + f(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) \\ &= \frac{(2fP + w_x)h^2 \Sigma}{2P\ell^2 r} + \frac{g^2 P(2f\Sigma + k^2 q)}{2w_u \Sigma} + f^2 - \frac{fg^2 P}{w_u} - \frac{fh^2 \Sigma}{\ell^2 r} \\ &= f^2 + \frac{w_x h^2 \Sigma}{2P\ell^2 r} + \frac{g^2 Pk^2 q}{2w_u \Sigma} \end{aligned}$$

Therefore,

$$\frac{g^2 p h^2 \Sigma}{w_u \ell^2 r} + f(f - \frac{g^2 p}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) > 0 \quad (113)$$

for all  $(f, g, h) \in \mathbb{R}^3$  with  $f \neq 0$ ,  $g \neq 0$ , or  $h \neq 0$ . Using (113) in (110) we have the result

$$g - g_d < 0 \quad (114)$$

since  $g < 0$ . Therefore,  $J(u_0, f, g_\theta, h)$  is continuous over  $(-\infty, g)$ . Now we take the limit of  $J(u_0, f, g_\theta, h)$  as  $g_\theta \rightarrow -\infty$ . Using Equation (107),

$$\begin{aligned} \lim_{g_\theta \rightarrow -\infty} J(u_0, f, g_\theta, h) &= \lim_{g_\theta \rightarrow -\infty} \frac{\frac{G_1}{g_\theta} + G_2 + G_3 g_\theta}{(f + A)(\frac{fA}{g_\theta} + G_4)} \\ &= \lim_{g_\theta \rightarrow -\infty} \frac{G_3 g_\theta}{(f + A)G_4} \end{aligned}$$

Using the definition of  $A$  in (101) we note that

$$f + A = f - \frac{g^2 p}{w_u} + f - \frac{h^2 \Sigma}{\ell^2 r} < 0 \quad (115)$$

since we know  $f - \frac{g^2 p}{w_u} < 0$  and  $f - \frac{h^2 \Sigma}{\ell^2 r} < 0$ . Therefore, using Equations (108c) and (108d), we note

$$\frac{G_3}{(f + A)} > 0$$

and

$$\frac{g_\theta}{G_4} > 0 \quad \text{for } g < 0 \text{ and } g_\theta \in (-\infty, g)$$

Therefore

$$\lim_{g_\theta \rightarrow -\infty} J(u_0, f, g_\theta, h) = +\infty \quad (116)$$

Since  $J(u_0, f, g_\theta, h)$  is increasing and continuous at  $(f, g, h)$ , then  $J(u_0, f, g_\theta, h) < J(u_0, f, g, h)$  for some neighborhood  $(g - \epsilon, g)$ ,  $\epsilon > 0$ . Since  $J(u_0, f, g_\theta, h)$  is continuous over  $(-\infty, g)$  and has limit  $+\infty$  as  $g_\theta \rightarrow -\infty$ , there exists some point  $g_{\max} \in (-\infty, g)$  such that  $J(u_0, f, g_{\max}, h) = J(u_0, f, g, h)$ . Therefore, Condition (I4) is true for  $g < 0$  and with  $\Theta_G$  given as in (77a).

Case (2):  $g > 0$ ,  $\Theta_G = \{g_\theta \in [g_a, g_b] \mid 0 < g_a < g_b\}$ .

We have shown that the maximin point  $g$  equals  $g_a$  in this case. Therefore, to prove Condition (I4) in this case, it is necessary to show that there exists a  $g_{\max} \in (g, \infty)$  such that  $J(u_0, f, g_{\max}, h) = J(u_0, f, g, h)$ . First, note that from Equation (84) and the subsequent discussion we have shown that  $J(u_0, f, g_\theta, h)$  is decreasing at  $(f, g, h)$ . Equation (109) gives the only point,  $g_d$ , where  $J(u_0, f, g_\theta, h)$  is discontinuous. Using the expression for  $g - g_d$  given in Equation (110), and the relation, (113), we obtain the result

$$g - g_d > 0$$

since  $g > 0$ . Therefore  $J(u_0, f, g_\theta, h)$  is continuous over  $(g, \infty)$ . Now we take the limit of  $J(u_0, f, g_\theta, h)$  as  $g_\theta \rightarrow \infty$ . Using Equation (107),

$$\begin{aligned} \lim_{g_\theta \rightarrow \infty} J(u_0, f, g_\theta, h) &= \lim_{g_\theta \rightarrow \infty} \frac{\frac{G_1}{g_\theta} + G_2 + G_3 g_\theta}{(f + A) \left( \frac{fA}{g_\theta} + G_4 \right)} \\ &= \lim_{g_\theta \rightarrow \infty} \frac{G_3 g_\theta}{(f + A) G_4} \end{aligned}$$

$$\lim_{g_\theta \rightarrow \infty} J(u_0, f, g_\theta, h) = +\infty \quad (117)$$

since  $\frac{G_3}{(f+A)} > 0$  and  $\frac{g_\theta}{G_4} > 0$  for  $g > 0$  and  $g_\theta \in (g, \infty)$ . Since  $J(u_0, f, g_\theta, h)$  is decreasing and continuous at  $(f, g, h)$ , then  $J(u_0, f, g_\theta, h) < J(u_0, f, g, h)$  for some neighborhood  $(g, g + \varepsilon)$ ,  $\varepsilon > 0$ . Since  $J(u_0, f, g_\theta, h)$  is continuous over  $(g, \infty)$ , and has limit  $+\infty$  as  $g_\theta \rightarrow \infty$ , there exists some point  $g_{\max} \in (g, \infty)$  such that  $J(u_0, f, g_{\max}, h) = J(u_0, f, g, h)$ . Therefore, Condition (I4) holds for  $g > 0$  and  $\Theta_C$  given in (77b).

We now compute  $g_{\max}$  using Equation (107) with (108).

$$J(u_0, f, g_{\max}, h) = J(u_0, f, g, h)$$

$$\frac{G_1 + G_2 g_{\max} + G_3 g_{\max}^2}{(f+A)(fA + G_4 g_{\max})} = \frac{G_1 + G_2 g + G_3 g^2}{(f+A)(fA + G_4 g)}$$

Cross-multiplying, and cancelling the term,  $f+A$ , we obtain

$$(G_1 + G_2 g_{\max} + G_3 g_{\max}^2)(fA + G_4 g) = (G_1 + G_2 g + G_3 g^2)(fA + G_4 g_{\max})$$

By expanding this and factoring out the already known factor,  $g_{\max} - g$ ,

$$(g_{\max} - g)[G_3 fA(g_{\max} + g) + G_3 G_4 g g_{\max} + G_2 fA - G_1 G_4] = 0$$

Setting the second factor equal to zero, and solving for  $g_{\max}$ ,

$$g_{\max} = - \frac{[G_3 fAg + G_2 fA - G_1 G_4]}{G_3 (fA + G_4 g)}$$

Using the definitions of  $A$  in (101) and  $G_1$  through  $G_4$  in (108), and using a few manipulations, we obtain the final result:

$$g_{\max} = \frac{-gf(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) + \frac{gPh^2 \Sigma}{w_x \ell^2 r} (f^2 + \frac{k^2 qh^2}{\ell^2 r}) + \frac{w_u k^2 q}{gP\Sigma} (f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})^2}{[f(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) + \frac{g^2 Ph^2 \Sigma}{w_u \ell^2 r}]} \quad (118)$$

Note (118) represents  $g_{\max}$  for both Case (1) and Case (2). This completes the consideration of Condition (I4).

Equation (104) evaluated at  $g_\theta = g$  is quadratic in  $f_\theta$ , and therefore has two solutions. One solution is  $f_\theta = f$ . Since this solution is real, and the coefficients of (104) are real, the other solution is also real, and Condition (I5) is true.

We now show that Condition (I6) is true. We start by investigating the asymptote equations, (105) and (106). Denote  $f_\theta$  in (105) by  $f_{d1}$ . Then, using (105)

$$f_{d1} + f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} = 0 \quad (119)$$

Subtracting (119) from Relation (115) we get the result

$$f < f_{d1} \quad \text{for all } g_\theta \in \Theta_G, \quad h_\theta \in \Theta_H \quad (120)$$

Denote  $f_\theta$  in (106) by  $f_{d2}$ . Then using (106)

$$f_{d2} = \frac{\frac{g_\theta g P h_\theta h \Sigma}{w_u \ell^2 r}}{-(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})} \quad (121)$$

We now show that  $f_{d2} > f$ . Comparing  $f_{d2}$  to  $f$  and using (121),

$$f_{d2} - f = \frac{\left[ \frac{g_\theta g P h_\Sigma}{w_u \ell^2 r} + f \left( f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} \right) \right]}{- \left( f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} \right)}$$

$$f_{d2} - f > \frac{\left[ \frac{g^2 P h^2 \Sigma}{w_u \ell^2 r} + f \left( f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} \right) \right]}{- \left( f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} \right)} \quad \text{for all } g_\theta \in \Theta_G, h_\theta \in \Theta_H$$

since  $g_\theta g > g^2$  and  $h_\theta h > h^2$  for all  $g_\theta \in \Theta_G$  and  $h_\theta \in \Theta_H$ . Therefore, since  $f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r} < 0$  and Relation (113) holds,  $f_{d2} - f > 0$ . That is,

$$f < f_{d2} \quad \text{for all } g_\theta \in \Theta_G, h_\theta \in \Theta_H. \quad (122)$$

Equation (121) evaluated at  $h_\theta = h$  has negative slope in the  $f_\theta$ - $g_\theta$  plane when  $g < 0$  and  $\Theta_G$  as in (77a), and positive slope in the  $f_\theta$ - $g_\theta$  plane if  $g > 0$  and  $\Theta_G$  as in (77b). The graphs of (119) and (121) are a pair of intersecting lines oriented as shown in Figure 15. From Figure 15 we can see that  $f_{d2} > f_{d1}$  for some  $g_\theta < g$ , when  $g < 0$  or for some  $g_\theta > g$  when  $g > 0$ . We will take the limit of  $J(u_o, f_\theta, g_\theta, h)$  as  $f_\theta \rightarrow f_{d2}^+$  at a value of  $g_\theta$  such that  $f_{d2} > f_{d1}$ , as shown in Figure 15. Examination of (99) shows that

$$\lim_{f_\theta \rightarrow f_{d2}^+} J(u_o, f_\theta, g_\theta, h) = \lim_{f_\theta \rightarrow f_{d2}^+} \frac{-1}{(f_\theta + A) \left( f_\theta A + \frac{g_\theta g h^2 \Sigma}{w_u \ell^2 r} \right)}$$

$$\left[ \frac{g^2 P^2 h^4 \Sigma^2 k^2 q}{2 w_u \ell^4 r^2} + \frac{g^2 P^2 h^2 \Sigma^2 w_x g_\theta^2}{2 w_u \ell^2 r} + \frac{f_\theta^2 g^2 P^2 h^2 \Sigma^2}{2 w_u \ell^2 r} + \frac{w_x k^2 q}{2} A^2 \right] \quad (123)$$



since the  $f_\theta$  term and the  $g_\theta h_\theta$  term (evaluated at  $h_\theta = h$ ) cancel when  $f_\theta = f_{d2}$ . Now since we are operating at  $f_\theta$ 's that are to the right of the asymptotes given by (119) and (121), the first term of the denominator of the right side of (123) is positive, where before it was negative, and the second term of the denominator is negative, where before it was positive. Therefore, the denominator is negative, and inspection of (123) shows that the numerator is also negative, so that  $J(u_o, f_\theta, g_\theta, h)$  is positive for values of  $f_\theta$  close to  $f_{d1}^+$ . Therefore,

$$\lim_{f_\theta \rightarrow f_{d1}^+} J(u_o, f_\theta, g_\theta, h) = +\infty \quad (124)$$

Now consider the portions of the lines (119) and (121) that are right-most which we have labeled  $f_d$  in Figure 15. Since  $J(u_o, f_\theta, g_\theta, h) = +\infty$  on the line segment corresponding to Equation (121), and  $J(u_o, f_\theta, g_\theta, h)$  is continuous for  $f_\theta > f_d$ ,  $J(u_o, f_\theta, g_\theta, h) = +\infty$  on the line segment corresponding to (119). Therefore,

$$\lim_{f_\theta \rightarrow f_d^+} J(u_o, f_\theta, g_\theta, h) = +\infty \quad \text{for all } g_\theta \in \mathbb{R} \quad (125)$$

We now take the limit of  $J(u_o, f_\theta, g, h)$  as  $f_\theta \rightarrow \infty$ . Equation (99) evaluated at  $g_\theta = g$  and  $h_\theta = h$  is of the form:

$$J(u_o, f_\theta, g, h) = \frac{F_1 + F_2 f_\theta + F_3 f_\theta^2}{(f_\theta + A)(f_\theta A + \frac{g^2 \text{Ph}^2 \Sigma}{w_u \ell^2 r})} \quad (126)$$

where

$$F_1 = - \left\{ \frac{g^2 P^2 h^4 \Sigma^2 k^2 q}{2w_u \ell^4 r^2} + \frac{g^4 P^2 h^2 \Sigma^2 w_x}{2w_u \ell^2 r^2} + \frac{g^2 P h^2 \Sigma}{2w_u \ell^2 r} (w_x k^2 q + \frac{g^2 P^2 h^2 \Sigma^2}{w_u \ell^2 r}) \right. \\ \left. + \frac{w_x k^2 q}{2} (f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})^2 \right\} \quad (127a)$$

$$F_2 = - \left\{ \frac{1}{2} (w_x k^2 q + \frac{g^2 P^2 h^2 \Sigma^2}{w_u \ell^2 r}) (f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) \right\} \quad (127b)$$

$$F_3 = - \left\{ \frac{g^2 P^2 h^2 \Sigma^2}{2w_u \ell^2 r} \right\} \quad (127c)$$

$$F_4 = \frac{g P h \Sigma}{w_u \ell^2 r} \quad (127d)$$

Taking the limit of  $J(u_o, f_\theta, g, h)$  using (126),

$$\lim_{f_\theta \rightarrow \infty} J(u_o, f_\theta, g, h) = \lim_{f_\theta \rightarrow \infty} \frac{\frac{F_1}{f_\theta^2} + \frac{F_2}{f_\theta} + F_3}{(1 + \frac{A}{f_\theta})(A + \frac{g^2 P h^2 \Sigma^2}{f_\theta w_u \ell^2 r})} = \frac{F_3}{A}$$

Using the definition of  $F_3$  in (127c) and  $A$  in (101),

$$\lim_{f_\theta \rightarrow \infty} J(u_o, f_\theta, g, h) = \frac{\frac{g^2 P^2 h^2 \Sigma^2}{2w_u \ell^2 r}}{(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})} \quad (128)$$

We now compare this value of  $J(u_o, f_\theta, g, h) \Big|_{f_\theta = \infty}$  with the maximin value given by Equation (49):

$$\begin{aligned}
J(u_o, f, g, h) - J(u_o, f_\theta, g, h) \Big|_{f_\theta = \infty} &= k^2_{qP} + \frac{g^2_{P\Sigma}}{w_u} + \frac{\frac{g^2_{P\Sigma} h^2_{\Sigma}}{2w_u \ell^2_r}}{(f - \frac{g^2_P}{w_u} - \frac{h^2_{\Sigma}}{\ell^2_r})} \\
&= k^2_{qP} + \frac{g^2_{P\Sigma} (f - \frac{g^2_P}{w_u} - \frac{h^2_{\Sigma}}{\ell^2_r})}{w_u (f - \frac{g^2_P}{w_u} - \frac{h^2_{\Sigma}}{\ell^2_r})}
\end{aligned}$$

Therefore,

$$J(u_o, f, g, h) - J(u_o, f_\theta, g, h) \Big|_{f_\theta = \infty} > 0 \quad (129)$$

But from Equation (125) we know

$$J(u_o, f, g, h) - J(u_o, f_\theta, g, h) \Big|_{f_\theta = f_d^+} < 0 \quad (130)$$

Therefore, since  $J(u_o, f_\theta, g, h)$  is continuous over  $(f_d, \infty)$ , there exists some point  $f_\theta = f_x$  in the interval  $(f_d, \infty)$  such that

$$J(u_o, f, g, h) - J(u_o, f_x, g, h) = 0$$

Therefore, Condition (I6) holds.

Condition (I6), supported by Condition (I5), shows that the graph (104) is not an ellipse since the graph of (104) would be required to cross the asymptotes that exist due to Condition (I2). Therefore, by Condition (I1) we know that the graph is a hyperbola. Therefore, Characteristic (Ia) is true. Conditions (I3) and (I4) are equivalent to Characteristic (Ib), and Conditions (I5) and (I6), supported by Condition (I2), are equivalent to Characteristic (Ic). This completes the proof of the validity of Characteristics (Ia), (Ib) and (Ic).

Characteristics (Ia), (Ib) and (Ic) enable us to construct the largest set in  $\chi$  with  $\Theta_H$  given in (78c). This set is shown in Figure 15 as the area with the darker shading.

Next we investigate the contour in the  $f_\theta - h_\theta$  plane with  $g_\theta = g$ . This situation is equivalent to setting  $\Theta_G$  as in (77c). Equation (102) then reduces to a quadratic equation of the form:

$$K_1 f_\theta^2 + K_2 f_\theta + K_3 g f_\theta h_\theta + K_4 g h_\theta + K_5 h_\theta^2 + K_6 g^2 + K_7 = 0 \quad (131)$$

A typical graph of Equation (104) is shown in Figure 16. This graph exhibits three basic characteristics we need in order to construct the largest set in  $\chi$  that enjoys the saddle-point property. These characteristics are:

- (IIa) The graph of (131) is a hyperbola.
- (IIb) Curve C intersects the line  $f_\theta = f$  at exactly one other point,  $h_\theta = h_{\max}$ , where  $h_{\max}$  is always in a set  $\Theta_H$  in  $\chi$ .
- (IIc) Curve C never intersects the line,  $h_\theta = h$ , at a point other than the maximin point,  $(f, g, h)$ .

We see that, as in the previous case, the shaded area to the left of Curve C is a convex set in the  $f_\theta - h_\theta$  plane, and is the largest set that enjoys the saddle-point property since we have already shown that  $J(u_0, f_\theta, g_\theta, h_\theta)$  is decreasing at  $(f, g, h)$  in all directions into the area with darker shading. However, this set is not rectangular and therefore is not in  $\chi$ .

Characteristics (IIa), (IIb) and (IIc) enable us to construct the largest set in  $\chi$  that enjoys the saddle-point property. This set is shown in Figure 16 with darker shading.

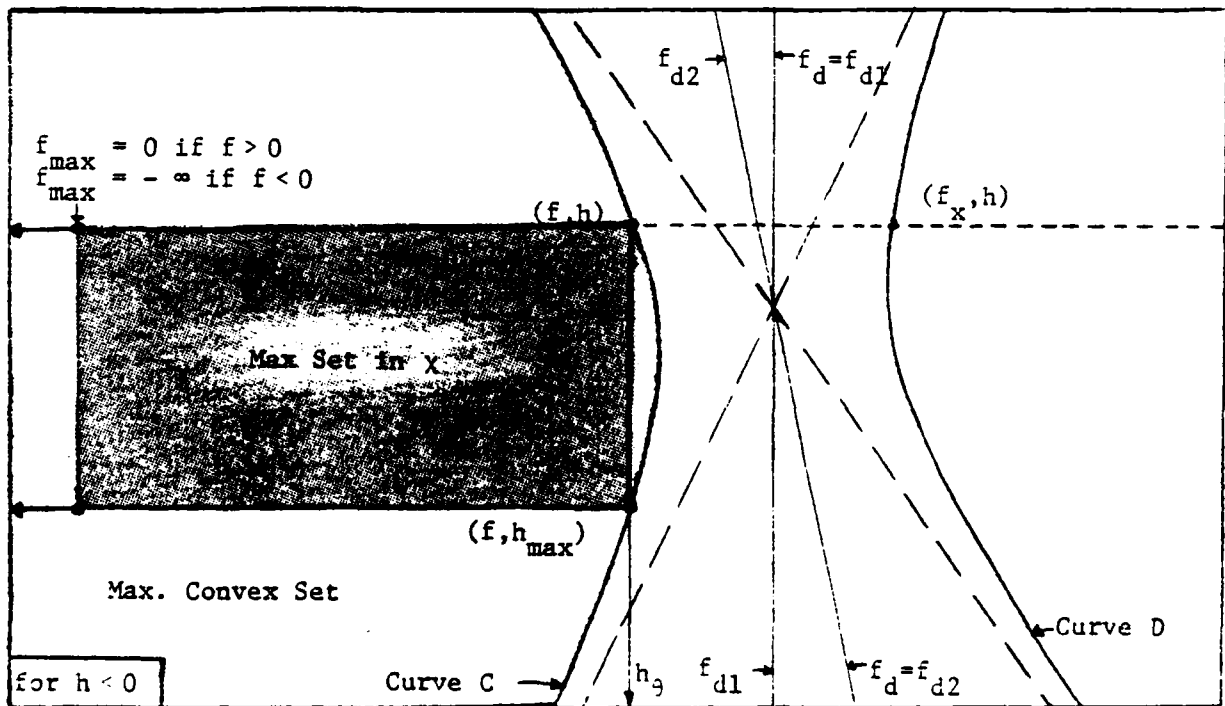
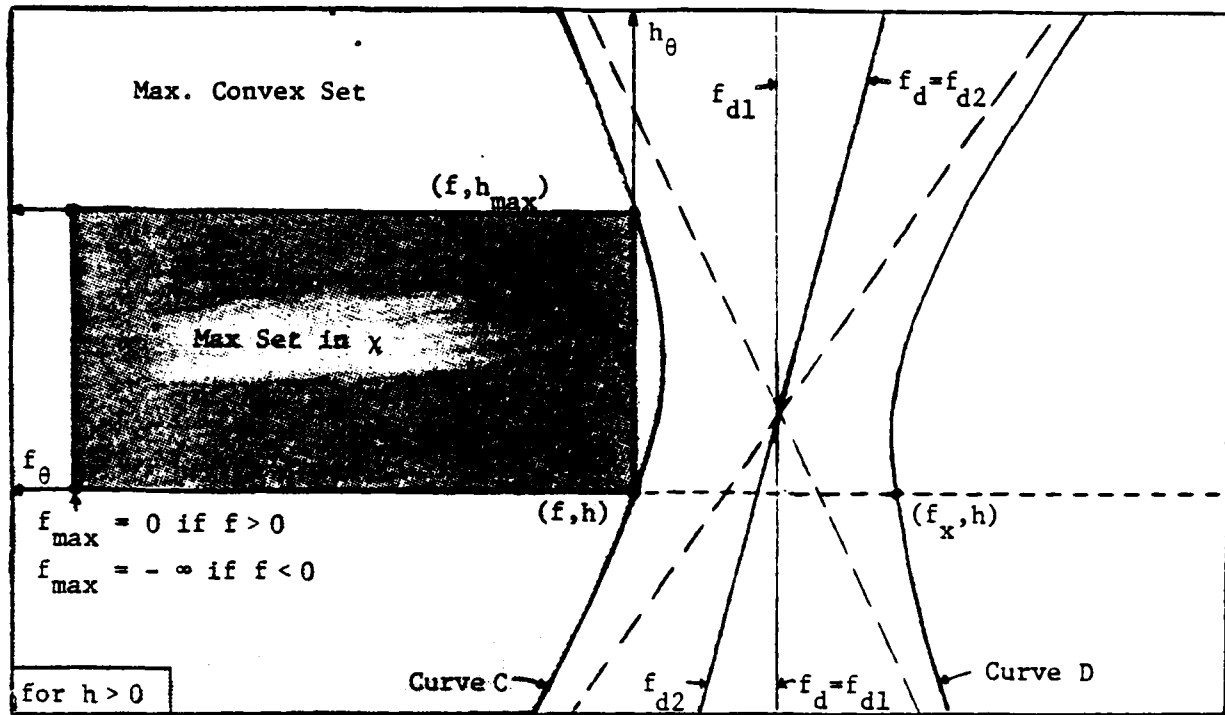


Figure 16. Contour of  $J(f_0, g, h_0) = J(f, g, h)$  in the  $f_0 - h_0$  plane.

It remains to show that Characteristics (IIa), (IIb) and (IIc) are valid. Necessary and sufficient conditions for (IIa), (IIb) and (IIc) to be true are:

- (II1) The graph of Equation (131) is either a hyperbola or an ellipse.
- (II2) There exist two asymptotes of  $J(u_0, f_\theta, g, h_\theta)$  in the  $f_\theta - h_\theta$  plane. They are a pair of intersecting lines.
- (II3) There always exists exactly one other point besides  $(f, g, h)$  such that  $J(u_0, f, g, h_\theta) = J(u_0, f, g, h)$ .
- (II4) This point is always in some set  $\Theta_H$  of the class,  $\chi$ , under consideration, and is the point,  $(f, g, h_{\max})$ , represented in Figure 16.
- (II5) There always exists exactly one other point besides  $(f, g, h)$  such that  $J(u_0, f_\theta, g, h) = J(u_0, f, g, h)$ .
- (II6) This point is always on the opposite side of the asymptotes of  $J(u_0, f_\theta, g, h_\theta)$ , and is the point  $(f_x, g, h)$  on Curve D in Figure 16.

Conditions (II1), (II2), (II5) and (II6) verify Characteristic (IIa). Conditions (II3) and (II4) verify Characteristic (IIb), and Conditions (II2), (II5) and (II6) verify Characteristic (IIc). We now show that Conditions (II1) through (II6) are valid, and discuss the precise reason why they prove that Characteristics (IIa), (IIb) and (IIc) are valid.

Since Equation (131) is quadratic in  $f_\theta$  and  $h_\theta$ , and since  $K_1 \neq 0$ ,  $K_5 \neq 0$ ,  $K_6 \neq 0$  and  $K_7 \neq 0$ , its graph is either a hyperbola or an ellipse. Therefore, Condition (II1) holds.

To prove Condition (II2) we again investigate the denominator of (99) by setting each factor equal to zero. From Equation (105) we see that one asymptote is a vertical line in the  $f_\theta - h_\theta$  plane. From Equation (106)

we see that with  $g_\theta = g$ , its graph is a line in the  $f_\theta - h_\theta$  plane. Since the slope of (106) is different than that of (105), these lines intersect at some point. Since the numerator of (99) is nonzero for values of  $f_\theta$  and  $h_\theta$  satisfying (105) and (106), Condition (II2) holds.

Equation (131) evaluated at  $f_\theta = f$  is quadratic in  $h_\theta$ , and therefore, has two solutions. One solution is  $h_\theta = h$ . Since this solution is real, and the coefficients of (131) are real, the other solution is also real, and Condition (II3) indeed holds.

Equation (99) evaluated at  $f_\theta = f$  and  $g_\theta = g$  is of the form:

$$J(u_o, f, g, h_\theta) = \frac{H_1 + H_2 h_\theta + H_3 h_\theta^2}{(f + A)(fA + G_4 h_\theta)} \quad (132)$$

where

$$H_1 = - \left\{ \frac{g^4 P^2 h^2 \Sigma^2 w_x^2}{2w_u^2 \ell^2 r} + \frac{f^2 g^2 P^2 h^2 \Sigma^2}{2w_u \ell^2 r} + \frac{f}{2} (w_x k^2 q + \frac{g^2 P^2 h^2 \Sigma^2}{w_u \ell^2 r}) (f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) + \frac{w_x k^2 q}{2} (f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})^2 \right\} \quad (133a)$$

$$H_2 = - \left\{ \frac{g^2 P h \Sigma}{2w_u \ell^2 r} (w_x k^2 q + \frac{g^2 P^2 h^2 \Sigma^2}{w_u \ell^2 r}) \right\} \quad (133b)$$

$$H_3 = - \left\{ \frac{g^2 P^2 h^2 \Sigma^2}{2w_u \ell^4 r^2} \right\} \quad (133c)$$

$$H_4 = \frac{g^2 P h \Sigma}{w_u \ell^2 r} \quad (133d)$$

We now show that Condition (II4) holds. We consider the cases,  $h < 0$  corresponding to  $\Theta_H$  in (78a) and  $h > 0$  corresponding to  $\Theta_H$  in (78b), separately.

Case (1):  $h < 0$   $\Theta_H = \{h_\theta \in [h_a, h_b] \mid h_a < h_b < 0\}$

We have shown that the minimax point,  $h$ , equals  $h_b$  in this case. Therefore, to prove the validity of Condition (II4) in this case, it is necessary to show that there exists a  $h_{\max} \in (-\infty, h)$  such that  $J(u_o, f, g, h_{\max}) = J(u_o, f, g, h)$ . First, note that from Equation (88) and the subsequent discussion, we have shown that  $J(u_o, f, g, h_\theta)$  is increasing at  $(f, g, h)$ . Now consider Equation (106), evaluated at  $f_\theta = f$  and  $g_\theta = g$ , which yields a value of  $h_\theta$  which we denote by  $h_d$ :

$$h_d = \frac{f(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})}{\frac{g^2 P h \Sigma}{w_u \ell^2 r}} \quad (134)$$

$(f, g, h_d)$  is the only point where  $J(u_o, f, g, h_\theta)$  is discontinuous in  $h_\theta$ . We show that  $h < h_d$  so that  $J(u_o, f, g, h_\theta)$  is continuous over  $(-\infty, h)$ . Note that

$$h - h_d = \frac{\left[ \frac{g^2 P h \Sigma}{w_u \ell^2 r} + f(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) \right]}{\frac{g^2 P h \Sigma}{w_u \ell^2 r}} \quad (135)$$

Using Equation (113) and the fact that  $h < 0$  we have the result

$$h - h_d < 0 \quad (136)$$

Therefore,  $J(u_o, f, g, h_\theta)$  is continuous over  $(-\infty, h)$ . Now we take the limit of  $J(u_o, f, g, h_\theta)$  as  $h_\theta \rightarrow -\infty$ . Using Equation (132),

$$\lim_{h_\theta \rightarrow -\infty} J(u_o, f, g, h_\theta) = \lim_{h_\theta \rightarrow -\infty} \frac{\frac{H_1}{h_\theta} + H_2 + H_3 h_\theta}{(f + A)(\frac{fA}{h_\theta} + H_4)} = \lim_{h_\theta \rightarrow -\infty} \frac{H_3 h_\theta}{(f + A)H_4}.$$



Using Relation (115) and the definitions of  $H_3$  in (133c) and  $H_4$  in (133d) we note

$$\frac{H_3}{(f + A)} > 0$$

and

$$\frac{h_\theta}{H_4} > 0 \quad \text{for } h < 0 \text{ and } h_\theta \in (-\infty, h) .$$

Therefore,

$$\lim_{h_\theta \rightarrow -\infty} J(u_0, f, g, h_\theta) = +\infty . \quad (137)$$

Since  $J(u_0, f, g, h_\theta)$  is increasing and continuous at  $(f, g, h)$ , then  $J(u_0, f, g, h_\theta) < J(u_0, f, g, h)$  for some neighborhood  $(h - \epsilon, h)$ ,  $\epsilon > 0$ . Since  $J(u_0, f, g, h_\theta)$  is continuous over  $(-\infty, h)$ , and has limit  $+\infty$  as  $h_\theta \rightarrow -\infty$ , there exists some point  $h_{\max} \in (-\infty, h)$  such that  $J(u_0, f, g, h_{\max}) = J(u_0, f, g, h)$ . Therefore, Condition (II4) is true for  $h < 0$  and with  $\Theta_H$  given as in (78a).

Case (2):  $h > 0 \quad \Theta_H = \{h_\theta \in [h_a, h_b] \quad 0 < h_a < h_b\}$

We have shown that the maximin point,  $h$ , equals  $h_a$  in this case. Therefore, to prove Condition (II4) in this case, it is necessary to show that there exists an  $h_{\max} \in (h, \infty)$  such that  $J(u_0, f, g, h_{\max}) = J(u_0, f, g, h)$ . First, note that from Equation (88) and the subsequent discussion, we have shown that  $J(u_0, f, g, h_\theta)$  is decreasing at  $(f, g, h)$ . Equation (134) gives the only point,  $h_d$ , where  $J(u_0, f, g, h_\theta)$  is discontinuous. Using the expression for  $h - h_d$  given in Equation (135), the Relation (113), and the fact that  $h > 0$ , we obtain the result:

$$h - h_d > 0$$

Therefore,  $J(u_o, f, g, h_\theta)$  is continuous over  $(h, \infty)$ . Now we take the limit of  $J(u_o, f, g, h_\theta)$  as  $h_\theta \rightarrow \infty$ . Using Equation (132),

$$\lim_{h_\theta \rightarrow \infty} J(u_o, f, g, h_\theta) = \lim_{h_\theta \rightarrow \infty} \frac{\frac{H_1}{h_\theta} + H_2 + H_3 h_\theta}{(f + A)(\frac{fA}{h_\theta} + H_4)} = \lim_{h_\theta \rightarrow \infty} \frac{H_3 h_\theta}{(f + A)H_4}$$

Therefore,

$$\lim_{h_\theta \rightarrow \infty} J(u_o, f, g, h_\theta) = +\infty \quad (138)$$

since  $\frac{H_3}{(f + A)} > 0$  and  $\frac{h_\theta}{H_4} > 0$  for  $h > 0$  and  $h_\theta \in (h, \infty)$ . Since  $J(u_o, f, g, h_\theta)$  is decreasing and continuous at  $(f, g, h)$ , then  $J(u_o, f, g, h_\theta) < J(u_o, f, g, h)$  for some neighborhood  $(h, h + \epsilon)$ ,  $\epsilon > 0$ . Since  $J(u_o, f, g, h_\theta)$  is continuous over  $(h, \infty)$ , and has limit  $+\infty$  as  $h_\theta \rightarrow \infty$ , there exists some point  $h_{\max} \in (h, \infty)$  such that  $J(u_o, f, g, h_{\max}) = J(u_o, f, g, h)$ . Therefore, Condition (II4) holds for  $h > 0$  and  $\Theta_H$  given by (78b).

We now compute  $h_{\max}$  using Equation (132) with (133).

$$J(u_o, f, g, h_{\max}) = J(u_o, f, g, h)$$

$$\frac{H_1 + H_2 h_{\max} + H_3 h_{\max}^2}{(f + A)(fA + H_4 h_{\max})} = \frac{H_1 + H_2 h + H_3 h^2}{(f + A)(fA + H_4 h)}$$

Cross-multiplying, and cancelling the term,  $f + A$ , we obtain

$$(H_1 + H_2 h_{\max} + H_3 h_{\max}^2)(fA + H_4 h) = (H_1 + H_2 h + H_3 h^2)(fA + H_4 h_{\max})$$

By expanding this and factoring out the already known factor,  $h_{\max} - h$ ,

$$(h_{\max} - h)[H_3 f A (h_{\max} + h) + H_3 H_4 h h_{\max} + H_2 f A - H_1 H_4] = 0$$

Setting the second factor equal to zero, and solving for  $h_{\max}$ ,

$$h_{\max} = \frac{-[H_3 f A h + H_2 f A - H_1 H_4]}{H_3 (f A + H_4 h)}$$

Using the definitions of A in (101), and  $H_1$  through  $H_4$  in (133), and using a few manipulations, we obtain the final result:

$$h_{\max} = \frac{-hf(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) + \frac{g^2 Ph \Sigma}{w_u k^2 q} (f^2 + \frac{g^2 w_x}{w_u}) + \frac{w_x \ell^2 r}{Ph \Sigma} (f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r})^2}{[f(f - \frac{g^2 P}{w_u} - \frac{h^2 \Sigma}{\ell^2 r}) + \frac{g^2 Ph^2 \Sigma}{w_u \ell^2 r}]}$$
(139)

Note that (139) represents  $h_{\max}$  for both Case (1) and Case (2). This completes the consideration of Condition (II4).

Note that Conditions (II5) and (II6) are equivalent to Conditions (I5) and (I6) which we have shown to be true. Therefore, Conditions (II5) and (II6) hold.

As in the previous case, Condition (II6), supported by Condition (II5), shows that the graph of (131) is not an ellipse since the graph of (131) would be required to cross the asymptotes that exist due to Condition (II2). Therefore, by Condition (II1), we know that the graph is a hyperbola. Therefore, Characteristic (IIa) is true. Conditions (II3) and (II4) are equivalent to Characteristic (IIb), and Conditions (II5) and (II6), supported by Condition (II2), are equivalent to Characteristic (IIc). This completes the proof of the validity of Characteristics (IIa), (IIb) and (IIc).

Characteristics (IIa), (IIb) and (IIc) enable us to construct the largest set in  $\chi$  with  $\Theta_G$  given in (77c). This set is shown in Figure 16 as the area with the darker shading.

Finally, we investigate the contour in the  $g_\theta - h_\theta$  plane with  $f_\theta = f$ . This situation is equivalent to taking  $\Theta_F$  as in (76c). Equation (102) then reduces to a quadratic equation of the form:

$$K_6 g_\theta^2 + (K_3 f + K_4) g_\theta h_\theta + K_5 h_\theta^2 + K_1 f^2 + K_2 f + K_7 = 0 \quad (140)$$

A typical graph of Equation (140) is shown in Figure 17. This graph exhibits three basic characteristics we need in order to construct the largest set in  $\chi$  that enjoys the saddle-point property. These characteristics are:

(IIIa) The graph of (140) is a hyperbola.

(IIIb) Curve E intersects the line,  $h_\theta = h$ , at exactly one other point

$$g_\theta = g_{\max} \text{ where } g_{\max} \text{ is always in a set } \Theta_G \text{ in } \chi.$$

(IIIc) Curve E intersects the line,  $g_\theta = g$ , at exactly one other point

$$h_\theta = h_{\max} \text{ where } h_{\max} \text{ is always in a set } \Theta_H \text{ in } \chi.$$

The shaded area to the left of Curve E is a convex set in the  $g_\theta - h_\theta$  plane, and is the largest set that enjoys the saddle-point property, since we have already shown that  $J(u_0, f_\theta, g_\theta, h_\theta)$  is decreasing at  $(f, g, h)$  in all directions into the area with darker shading. However, this set is not rectangular and therefore is not in  $\chi$ .

Characteristics (IIIa), (IIIb) and (IIIc) enable us to construct the largest set in  $\chi$  that enjoys the saddle-point property. This set is shown in Figure 17 with darker shading.

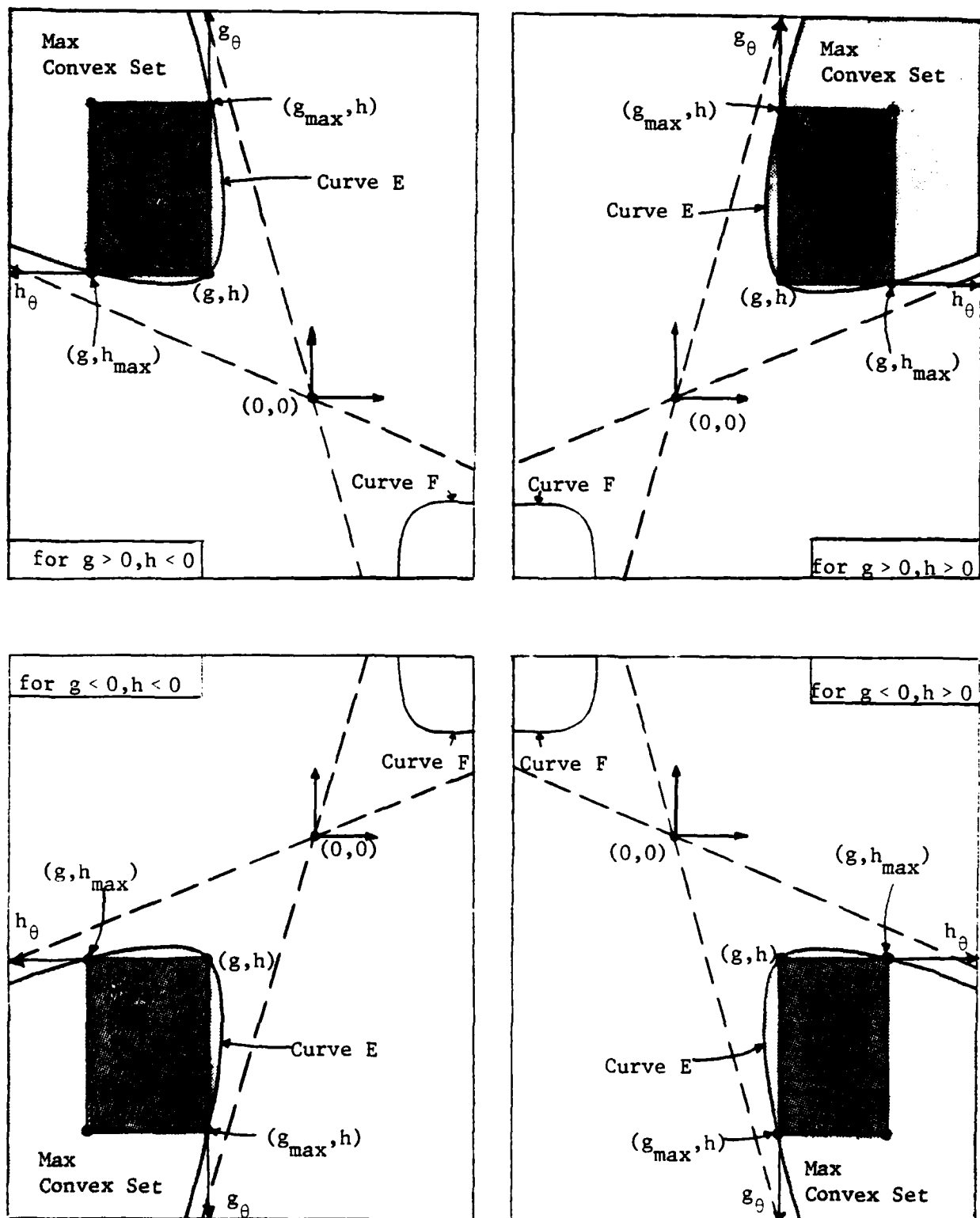


Figure 17. Contour of  $J(f, g_\theta, h_\theta) = J(f, g, h)$  in the  $h_\theta - g_\theta$  plane.

It remains to show that Characteristics (IIIa), (IIIb) and (IIIc) are valid. Necessary and sufficient conditions for (IIIa), (IIIb) and (IIIc) to be true are:

- (III1) The graph of (140) is either a hyperbola or an ellipse.
- (III2) The graph of (140) has center at the origin.
- (III3) There always exists exactly one other point besides  $(f, g, h)$  such that  $J(u_0, f, g_0, h) = J(u_0, f, g, h)$ .
- (III4) This point is always in some set  $\Theta_G$  of the class,  $\chi$ , under consideration. This is point  $(f, g_{\max}, h)$  as represented in Figure 17.
- (III5) There always exists exactly one other point besides  $(f, g, h)$  such that  $J(u_0, f, g, h_0) = J(u_0, f, g, h)$ .
- (III6) This point is always in some set  $\Theta_H$  of the class,  $\chi$ , under consideration. This is point  $(f, g, h_{\max})$  as represented in Figure 17.

Conditions (III1) through (III6) verify Characteristic (IIIa). Conditions (III3) and (III4) verify Characteristic (IIIb), and Conditions (III5) and (III6) verify Characteristic (IIIc). We now show that Conditions (III1) through (III6) are valid, and discuss the precise reasons why they prove that Characteristics (IIIa), (IIIb) and (IIIc) are valid.

Since Equation (140) is quadratic in  $g_0$  and  $h_0$ , and since  $K_1 f^2 + K_2 f + K_7 \neq 0$ ,  $K_5 \neq 0$  and  $K_6 \neq 0$ , its graph is either a hyperbola or a ellipse. Therefore, Condition (III1) is true.

Since there are no terms in (140) that are linear in  $g_0$ , and there are no terms that are linear in  $h_0$ , the graph of Equation (140) has center at the origin. Therefore, Condition (III2) is true.

Conditions (III3) and (III4) are equivalent to Conditions (I3) and (I4) which are known to be true. Therefore, Conditions (III3) and (III4) are true.

Conditions (III5) and (III6) are equivalent to Conditions (II3) and (II4), which are known to be true. Therefore, Conditions (III5) and (III6) are true.

Conditions (III3) through (III6) establish the existence of points  $(f, g_{\max}, h)$  and  $(f, g, h_{\max})$  as solutions of (140). We also know that  $(f, g, h)$  solves (140). Since, by Condition (III2), we know the graph has center at the origin, the graph of (140) cannot be an ellipse since we cannot find an ellipse centered at the origin that intersects these three points, as we can easily see from Figure 17. Therefore, by Condition (III1), the graph of (140) is a hyperbola and Characteristic (IIIa) is true. Conditions (III3) and (III4) are equivalent to Characteristic (IIIb), and Conditions (III5) and (III6) are equivalent to Characteristic (IIIc). This completes the proof of Characteristics (IIIa), (IIIb) and (IIIc).

Characteristics (IIIa), (IIIb) and (IIIc) enable us to construct the largest set in  $\chi$  with  $\Theta_F$  given in (76c). This set is shown as the area in Figure 17 with the darker shading.

We can combine the results that were arrived at through the study of the contours in the  $f_\theta - g_\theta$  plane, the  $f_\theta - h_\theta$  plane, and the  $g_\theta - h_\theta$  plane at values  $h_\theta = h$ ,  $g_\theta = g$  and  $f_\theta = f$ , respectively, to find the largest set in the class,  $\chi$ , when all three parameters are unknown. All parameter sets that are subsets of this set enjoy the saddle-point property. The result is summarized in the following theorem.

Theorem 3: Consider a subclass  $\mathcal{S}$  of the class,  $\chi$ , such that:

$$\mathcal{S} = \{\Theta : \Theta_{FS} \cup \Theta_{GS} \cup \Theta_{HS}\}$$

and  $\Theta_{FS}$ ,  $\Theta_{GS}$ ,  $\Theta_{HS}$  are each one of the following, respectively:

$$\Theta_{FS} = \{f_{\theta} \in [f_a, f] \mid -\infty < f_a \leq f < 0\} \quad (141a)$$

$$\text{or} \quad \Theta_{FS} = \{f_{\theta} \in [f_a, f] \mid 0 < f_a \leq f\} \quad (141b)$$

$$\text{or} \quad \Theta_{FS} = \{f_{\theta} = f \mid f \in \mathbb{R} \quad f \neq 0\} \quad (141c)$$

$$\Theta_{GS} = \{g_{\theta} \in [g_a, g] \mid g_{\max} \leq g_a \leq g < 0\} \quad (142a)$$

$$\text{or} \quad \Theta_{GS} = \{g_{\theta} \in [g, g_b] \mid 0 < g \leq g_b \leq g_{\max}\} \quad (142b)$$

$$\text{or} \quad \Theta_{GS} = \{g_{\theta} = g \mid g \in \mathbb{R} \quad g \neq 0\} \quad (142c)$$

$$\Theta_{HS} = \{h_{\theta} \in [h_a, h] \mid h_{\max} \leq h_a \leq h < 0\} \quad (143a)$$

$$\text{or} \quad \Theta_{HS} = \{h_{\theta} \in [h, h_b] \mid 0 < h \leq h_b \leq h_{\max}\} \quad (143b)$$

$$\text{or} \quad \Theta_{HS} = \{h_{\theta} = h \mid h \in \mathbb{R} \quad h \neq 0\} \quad (143c)$$

where  $g_{\max}$  is given by Equation (118), and  $h_{\max}$  is given by (139). The following results hold:

- 1)  $(f, g, h)$  is the solution to the maximin problem, (9), for scalar systems of the form described by (3), (4) and (5) for  $\Theta \in \mathcal{S}$ .
- 2) The minimax problem, (8), and the maximin problem, (9), for scalar systems of the form described by (3), (4) and (5) have the same solution if and only if  $\Theta$  belongs to the class,  $\mathcal{S}$ , described above.

This theorem describes all the possible parameter sets in the class  $\chi$  such that a saddle point exists at the maximin solution,  $(f, g, h)$ . However,



this theorem does not say that there is no minimax solution for sets in  $\chi$  that are not in  $\mathcal{S}$ , but only that the minimax solutions for these sets do not correspond to the maximin solutions.

This completes our solution to the minimax problem for scalar systems by the method of solving the maximin solution for the same system. We now consider the examples of the previous section, and compare the performance of the minimax controller to that of the least sensitive controller.

#### 4.4. Examples

The following scalar system was considered in Section 3, which we state here using the present notation for the uncertain parameters:

$$dx_t = f_\theta x_t dt + g_\theta u_t dt + dw_t \quad (32a)$$

$$dy_t = x_t dt + dv_t \quad (32b)$$

$$E \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad E \begin{bmatrix} \begin{pmatrix} dw_t \\ dv_t \end{pmatrix} \begin{pmatrix} dw_t \\ dv_t \end{pmatrix}^T \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} dt$$

$$J(u_0, f_\theta, g_\theta, 1) = E_{w,v} \left[ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x_t^2 + u_t^2) dt \right] \quad (33)$$

Here  $f_\theta = \theta_1$  and  $g_\theta = \theta_2$ . In Example 4.1,  $g_\theta$  is known and equal to 1, and  $f_\theta$  is unknown. In Example 4.2,  $f_\theta = 1$  and  $g_\theta$  is unknown.

For Example 4.1,  $f_\theta$  unknown and  $g_\theta = 1$ , we introduced a uniform distribution on the following sets in Example 3.1 of Chapter 3:

$$\Theta_{F1} = \{f_{\theta} \in [0.9, 1.1]\} \quad (144a)$$

$$\Theta_{F2} = \{f_{\theta} \in [0.7, 1.3]\} \quad (144b)$$

$$\Theta_{F3} = \{f_{\theta} \in [0.5, 1.5]\} \quad (144c)$$

For Example 4.2,  $f_{\theta} = 1$  and  $g_{\theta}$  unknown, we introduced a uniform distribution on the following sets in Example 3.2 of Chapter 3:

$$\Theta_{G1} = \{g_{\theta} \in [0.9, 1.1]\} \quad (145a)$$

$$\Theta_{G2} = \{g_{\theta} \in [0.7, 1.3]\} \quad (145b)$$

$$\Theta_{G3} = \{g_{\theta} \in [0.5, 1.5]\} \quad (145c)$$

Equations (47) represent the controller corresponding to the maximin solution,  $(f, g)$ , which is yet to be determined. For these examples Equations (47) reduce to:

$$u_0 = -g^P \hat{x}_t \quad (146a)$$

$$d\hat{x}_t = (f - g^2 P) \hat{x}_t dt + \Sigma(dy_t - \hat{x}_t dt) \quad (146b)$$

$$2fP - g^2 P^2 + 1 = 0 \quad (146c)$$

$$2f\Sigma - \Sigma^2 + 1 = 0 \quad (146d)$$

Consider Example 4.1 with  $g_{\theta} = g = 1$ , and the parameter sets in (144). We can use the Fréchet differentials of Theorem 2 to find the maximin solution for each of these sets. The result for each set is stated in Part (1) of Theorem 3. The maximin solutions are:

$$\Theta_{F1} : f = 1.1 \quad (147a)$$

$$\Theta_{F2} : f = 1.3 \quad (147b)$$

$$\Theta_{F3} : f = 1.5 \quad (147c)$$

UNCLASSIFIED

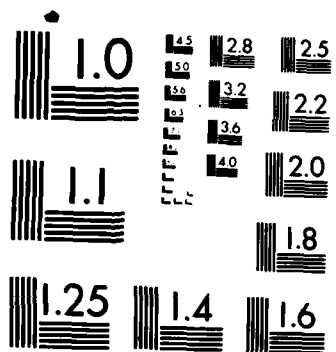
DECISION AND CONTROL LAB P H MCPWELL MAR 85 DC-77

212

44

F/G 12/1

END



MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

From Theorem 3, Part (2), we know the sets  $\Theta_{F1}$ ,  $\Theta_{F2}$  and  $\Theta_{F3}$  all enjoy the saddle-point property, and thus Solutions (147) are minimax solutions, and the corresponding controllers, (146), are minimax controllers.

We wish to investigate the performance of these minimax controllers as the parameter,  $f_\theta$ , varies in the sets  $\Theta_{F1}$ ,  $\Theta_{F2}$  and  $\Theta_{F3}$ . To facilitate this, we define  $J(u_\theta, f_\theta, g_\theta, 1)$  in Equation (99) by the notation  $J_{MM}(f_\theta, g_\theta)$ . We use Equation (99) to calculate  $J_{MM}$  for each example. Then the performance measure used in Chapter 3 is used, that is, a graph of  $J_{MM} - J_{OPT}$  as  $f_\theta$  or  $g_\theta$  varies in their respective sets.

To evaluate Equation (99) for a particular value of  $f$ , Equations (146c) and (146d) must be solved to find the corresponding  $P$  and  $\Sigma$ :

$$P = f + \sqrt{f^2 + 1} \quad (148a)$$

$$\Sigma = f + \sqrt{f^2 + 1} \quad (148b)$$

Therefore, for  $f = 1.1$ , and  $\Theta_{F1}$  as in (144a),

$$P = 2.5866$$

$$\Sigma = 2.5866$$

Then, Equation (99) reduces to:

$$J_{MM}(f_\theta) = \frac{22.3814 f_\theta^2 - 95.4886 f_\theta + 206.5586}{(4.1732 - f_\theta)(6.6905 - 4.1732 f_\theta)} \quad (149)$$

A plot of  $J_{MM}(f_\theta) - J_{OPT}(f_\theta)$  over  $[0.9, 1.1]$  is shown in Figure 18.

For  $f = 1.3$ , and  $\Theta_{F2}$  in (144b),

$$P = 2.9401$$

$$\Sigma = 2.9401$$

Then, Equation (99) reduces to:

$$J_{MM}(f_\theta) = \frac{37.3610 f_\theta^2 - 173.4110 f_\theta + 412.4891}{(4.5802 - f_\theta)(8.6442 - 4.5802 f_\theta)} \quad (150)$$

A plot of  $J_{MM}(f_\theta) - J_{OPT}(f_\theta)$  over  $[0.7, 1.3]$  is shown in Figure 19.

For  $f = 1.5$ ,

$$P = 3.3028$$

$$\Sigma = 3.3028$$

Then Equation (99) reduces to:

$$J_{MM}(f_\theta) = \frac{59.4976 f_\theta^2 - 306.3235 f_\theta + 786.5119}{(5.1056 - f_\theta)(10.9085 - 5.1056 f_\theta)} \quad (151)$$

A plot of  $J_{MM}(f_\theta) - J_{OPT}(f_\theta)$  over  $[0.5, 1.5]$  is shown in Figure 20.

Figures 18, 19 and 20 show the relative cost of the minimax controllers. Also, these figures show the relative cost of the least sensitive controllers we previously displayed in Figures 1, 2 and 3. From these plots, we can compare the performance of the minimax controllers with the performance of the least sensitive controllers. Several observations can be made.

- 1) Recall that the minimax controllers are optimal at the worst-case parameter value, whereas the least sensitive controllers are not optimal at any parameter value in the intervals under consideration.
- 2) The least sensitive controllers exhibit superior performance over a wider range of parameter values than the minimax controllers exhibit. This property can be quantified as the percentage of full range that a particular controller exhibits superior performance.

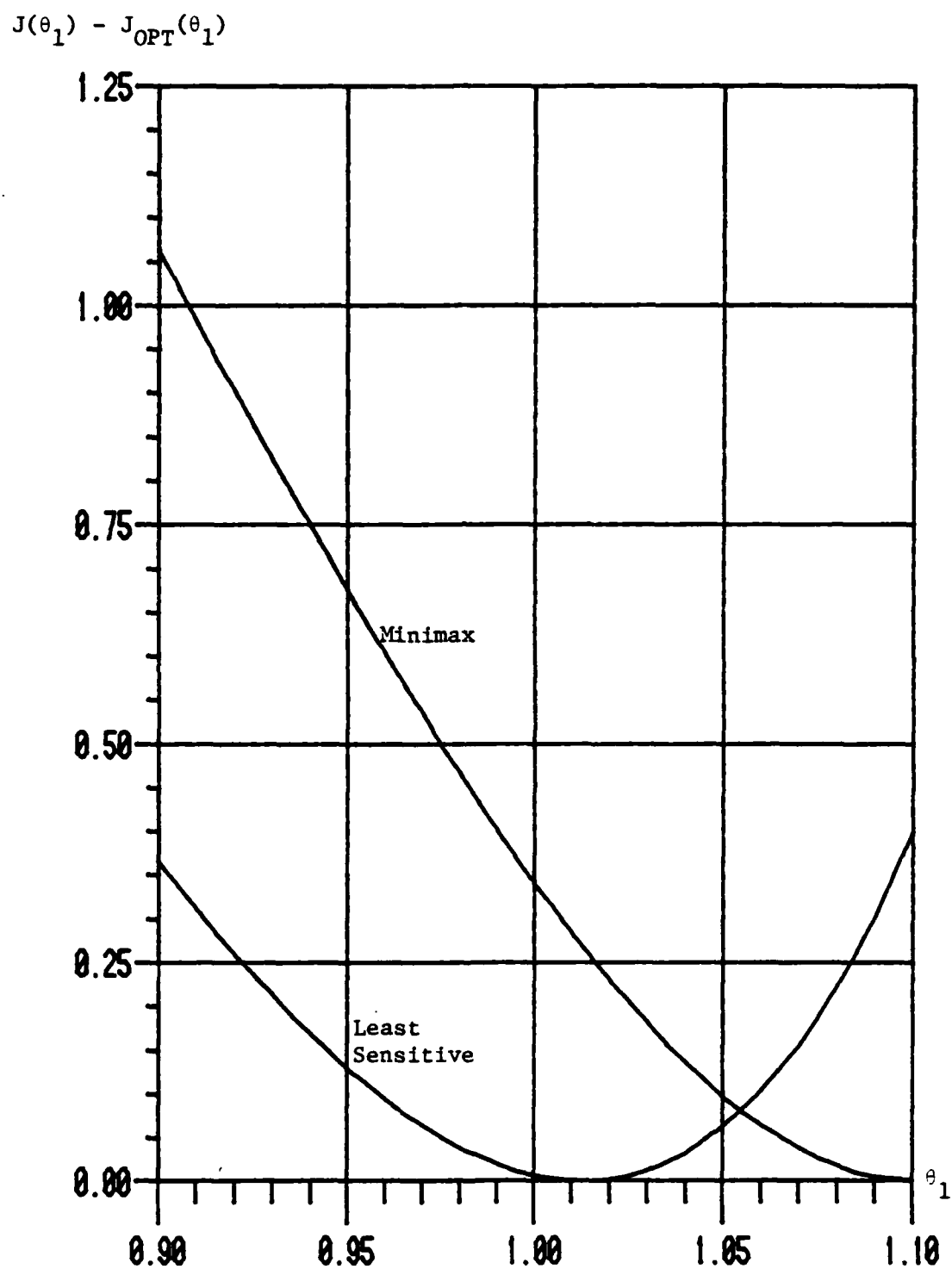


Figure 18.  $J_{\text{MM}} - J_{\text{OPT}}$  and  $J_{\text{LS}} - J_{\text{OPT}}$  vs.  $\theta_1$ ,  $\theta_2 = 1$ ,  $\theta_1 \in [0.9, 1.1]$ .

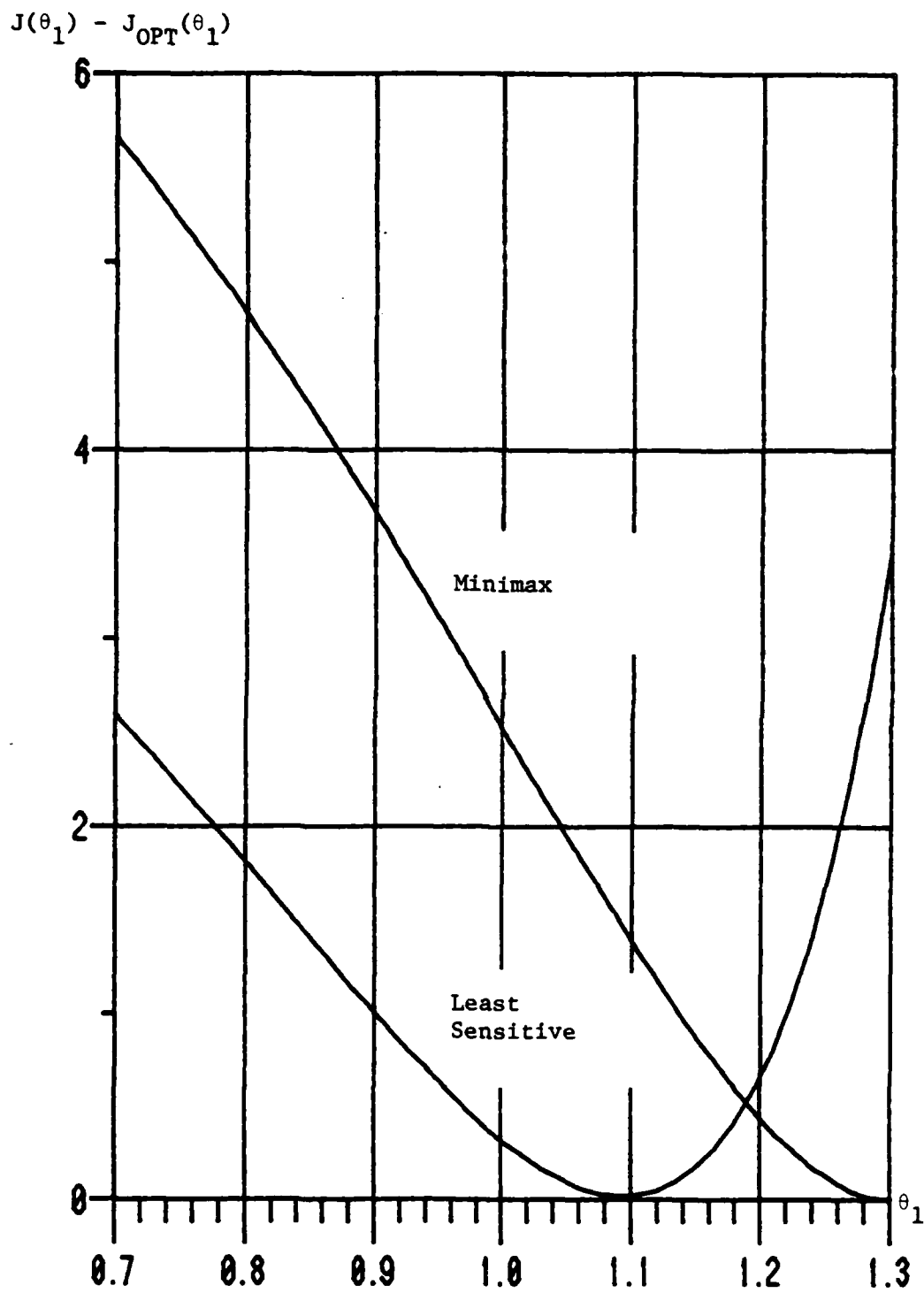


Figure 19.  $J_{MM} - J_{OPT}$  and  $J_{LS} - J_{OPT}$  vs.  $\theta_1$ ,  $\theta_2 = 1$ ,  $\theta_1 \in [0.7, 1.3]$ .



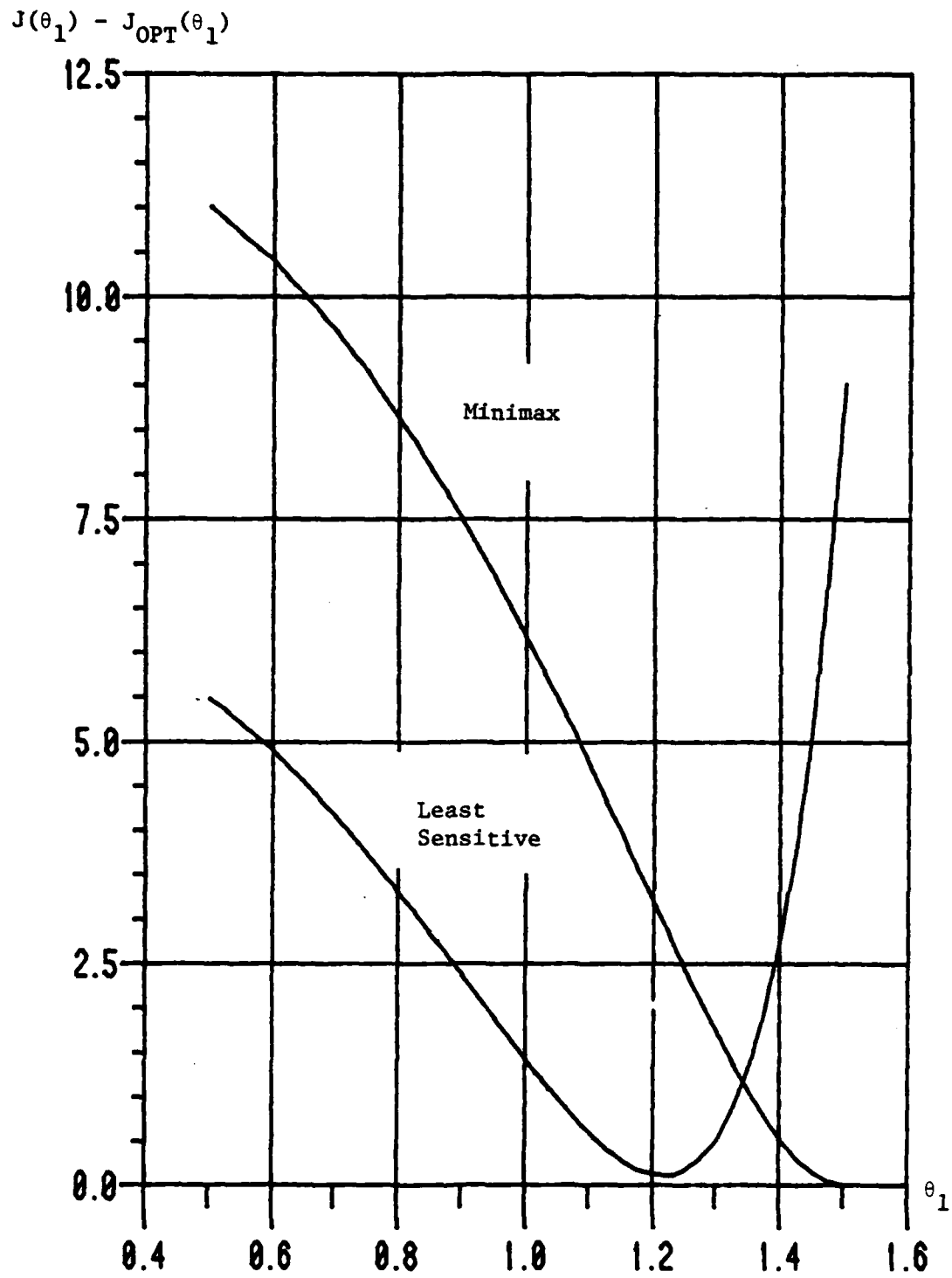


Figure 20.  $J_{MM} - J_{OPT}$  and  $J_{LS} - J_{OPT}$  vs.  $\theta_1$ ,  $\theta_2 = 1$ ,  $\theta_1 \in [0.5, 1.5]$ .

$\Theta_F$	% range (least sensitive)	% range (minimax)
[0.9,1.1]	77.5%	22.5%
[0.7,1.3]	82.3%	17.7%
[0.5,1.5]	84.2%	15.8%

From these figures we can see that the least sensitive controllers exhibit superior performance for an increasing relative range of parameter values as less parameter sensitivity is desired.

3) The relative difference between the maximum relative cost for the minimax controllers and the least sensitive controllers decreases as less sensitivity is desired. The relative maxima as a percentage of  $J_{MM}^{(max)} - J_{OPT}^{(max)}$  are:

$\Theta_F$	% difference
[0.9,1.1]	62.5%
[0.7,1.3]	40.9%
[0.5,1.5]	17.0%

Note that, as will be seen in the second example, this trend will continue. As we widen the range of parameter values, the maximum relative cost of the minimax controller is actually less than the maximum relative cost of the least sensitive controller.

These observations are useful when considering which design is best for a particular application. However, discussion of design considerations is postponed until after the investigation of Example 4.2.

Now consider Example 4.2 with  $f_0 = f = 1$  and the parameter sets in (145). We can use the Fréchet differentials of Theorem 2 to find the maximin solution for each of these sets. The result for each set is stated in Part (1) of Theorem 3. The maximin solutions are

$$\Theta_{G1} : g = 0.9 \quad (152a)$$

$$\Theta_{G2} : g = 0.7 \quad (152b)$$

$$\Theta_{G3} : g = 0.5 \quad (152c)$$

We now calculate  $g_{\max}$  for each of these sets using Equation (118). To evaluate (118) we need the corresponding  $P$  and  $\Sigma$  for each  $g$  in (152). From Equations (146c) and (146d) we can evaluate  $P$  and  $\Sigma$ :

$$P = \frac{1 + \sqrt{1 + g^2}}{g^2} \quad (153a)$$

$$\Sigma = 1 + \sqrt{2} \quad (153b)$$

Using Equations (118) to evaluate  $g_{\max}$ , the results are:

$$\begin{aligned} \Theta_{G1} : g = 0.9 \quad g_{\max} &= 9.5726 \\ \Theta_{G2} : g = 0.7 \quad g_{\max} &= 11.3463 \\ \Theta_{G3} : g = 0.5 \quad g_{\max} &= 14.8239 \end{aligned}$$

Therefore, by Theorem 3, Part (2), the sets  $\Theta_{G1}$ ,  $\Theta_{G2}$  and  $\Theta_{G3}$  all enjoy the saddle-point property, and thus Solutions (152) are minimax solutions, and the corresponding controllers, (146), are the minimax controllers.

It is interesting to note that, for this example, we enjoy a rather liberal choice of parameter sets in the sense that as the deviation from the mean increases, the upper bound on the set also increases, so that we are limited by the lower bound,  $g_0 = 0$ , rather than the upper bound,  $g_0 = g_{\max}$ . Therefore, for this example, all the desired deviations from the mean, up to 100%, have minimax controllers that we can design using the method described in this section.

To evaluate  $J_{MM}(g_\theta)$ , Equation (99) is used with  $P$  and  $\Sigma$  for each  $g$  given by Equations (153a) and (153b).

For  $g = 0.9$ , and  $\Theta_{G1}$  in (145a),

$$P = 2.8955$$

$$\Sigma = 2.4142 \quad .$$

Then Equation (99) reduces to:

$$J_{MM}(g_\theta) = \frac{19.7902 g_\theta^2 + 127.6514 g_\theta - 29.6352}{2.7596 (3.7596 - 6.2913 g_\theta)} \quad . \quad (154)$$

A plot of  $J_{MM}(g_\theta) - J_{OPT}(g_\theta)$  over  $[0.9, 1.1]$  is shown in Figure 21.

For  $g = 0.7$ , and  $\Theta_{G2}$  in (145b),

$$P = 4.5320$$

$$\Sigma = 2.4142 \quad .$$

Then Equation (99) reduces to:

$$J_{MM}(g_\theta) = \frac{29.3287 g_\theta^2 + 224.6223 g_\theta - 43.1607}{2.6349 (3.6349 - 7.6588 g_\theta)} \quad . \quad (155)$$

A plot of  $J_{MM}(g_\theta) - J_{OPT}(g_\theta)$  over  $[0.7, 1.3]$  is shown in Figure 22.

For  $g = 0.5$ , and  $\Theta_{G3}$  in (145c),

$$P = 8.4721$$

$$\Sigma = 2.4142 \quad .$$

Then Equation (99) reduces to:

$$J_{MM}(g_\theta) = \frac{52.2926 g_\theta^2 + 539.8936 g_\theta - 75.6519}{2.5322 (3.5322 - 10.2267 g_\theta)} \quad (156)$$

A plot of  $J_{MM}(g_\theta) - J_{OPT}(g_\theta)$  over  $[0.5, 1.5]$  is shown in Figure 23.

Figures 21, 22 and 23 show the relative cost of the minimax controllers. Also these figures show the relative cost of the least sensitive controllers that are displayed in Figures 8, 9 and 10. From these plots, similar observations can be made as in the previous example.

- 1) Again, the minimax controllers are optimal at the worst-case parameter value, and the least sensitive controllers are not optimal for any value.
- 2) The least sensitive controllers exhibit superior performance over a wider range of parameter values. The percent of full range that a particular controller exhibits superior performance is shown below:

$\theta_G$	% range (least sensitive)	% range (minimax)
[0.9, 1.1]	78.55%	21.45%
[0.7, 1.3]	85.93%	14.07%
[0.5, 1.5]	90.91%	9.09%

Again the least sensitive controllers exhibit superior performance for an increasing percentage of the full range of parameter values as less parameter sensitivity is desired.

- 3) The relative difference between the maximum relative cost for the minimax controllers and the least sensitive controllers decreases as less parameter sensitivity is desired. The relative maxima as a percentage of

$J_{MM}(\max) - J_{LS}(\max)$  are:

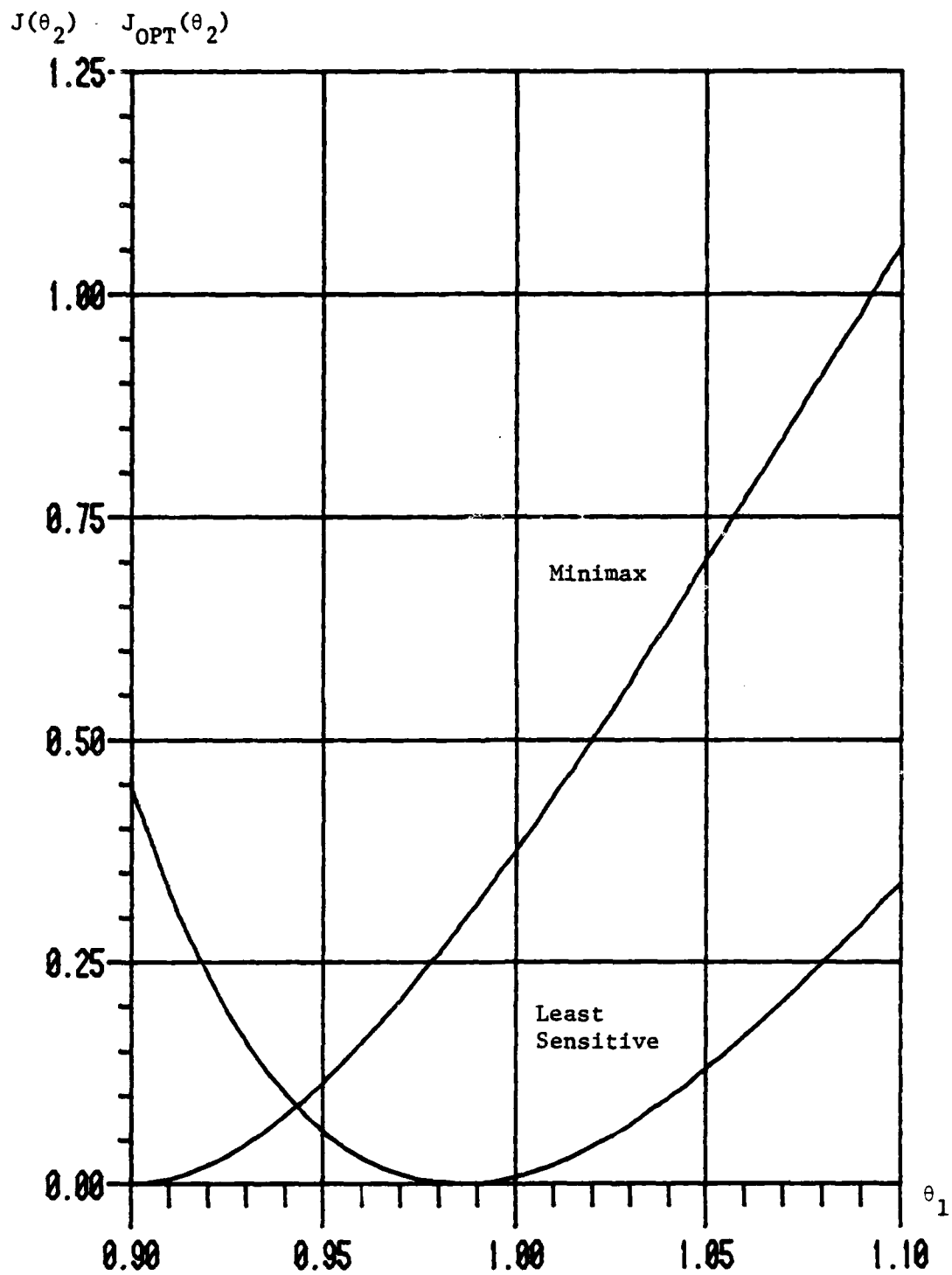


Figure 21.  $J_{MM} - J_{OPT}$  and  $J_{LS} - J_{OPT}$  vs.  $\theta_2$ ,  $\theta_1 = 1$ ,  $\theta_2 \in [0.9, 1.1]$ .

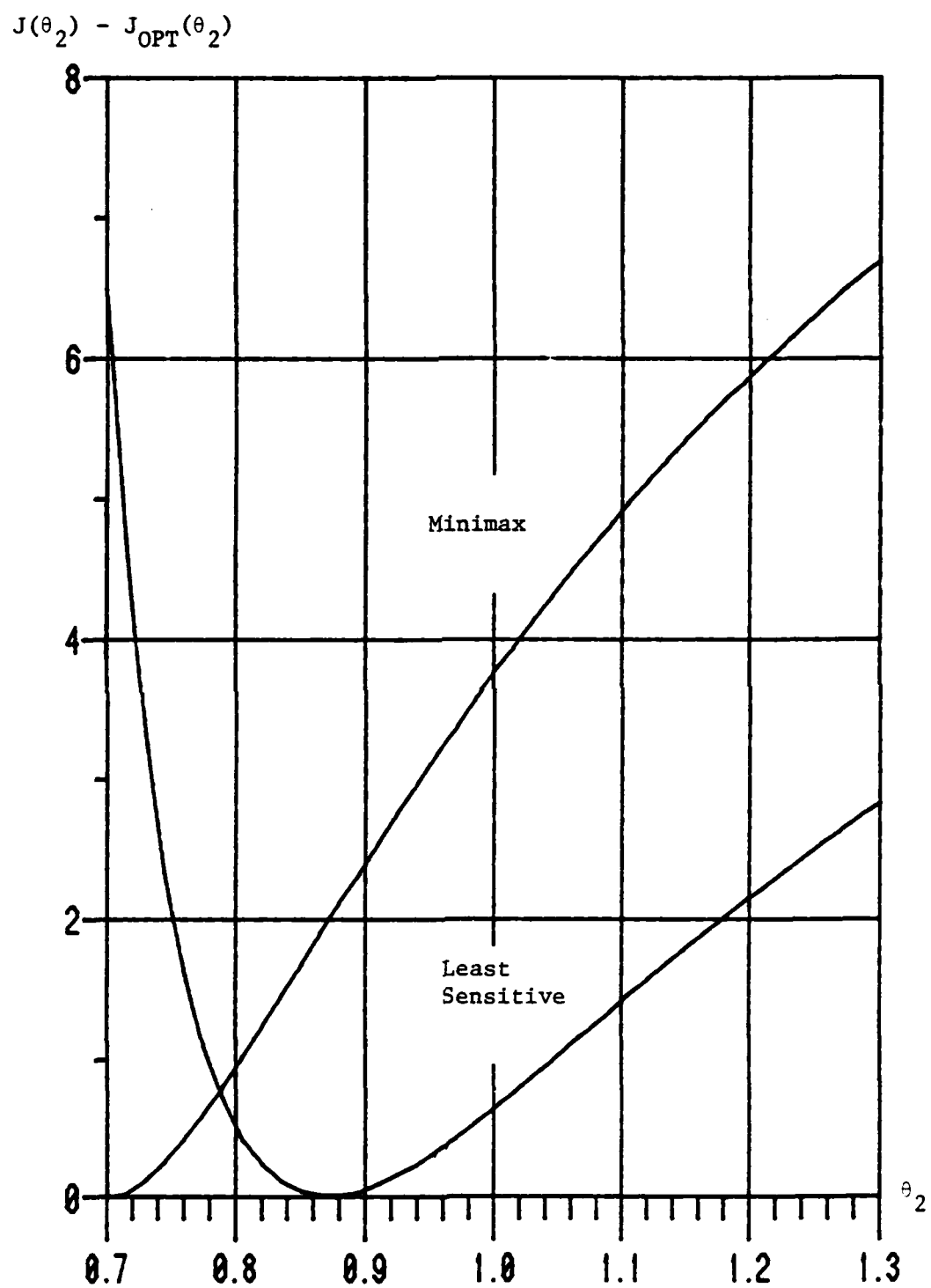


Figure 22.  $J_{MM} - J_{OPT}$  and  $J_{LS} - J_{OPT}$  vs.  $\theta_2$ ,  $\theta_1 = 1$ ,  $\theta_2 \in [0.7, 1.3]$ .

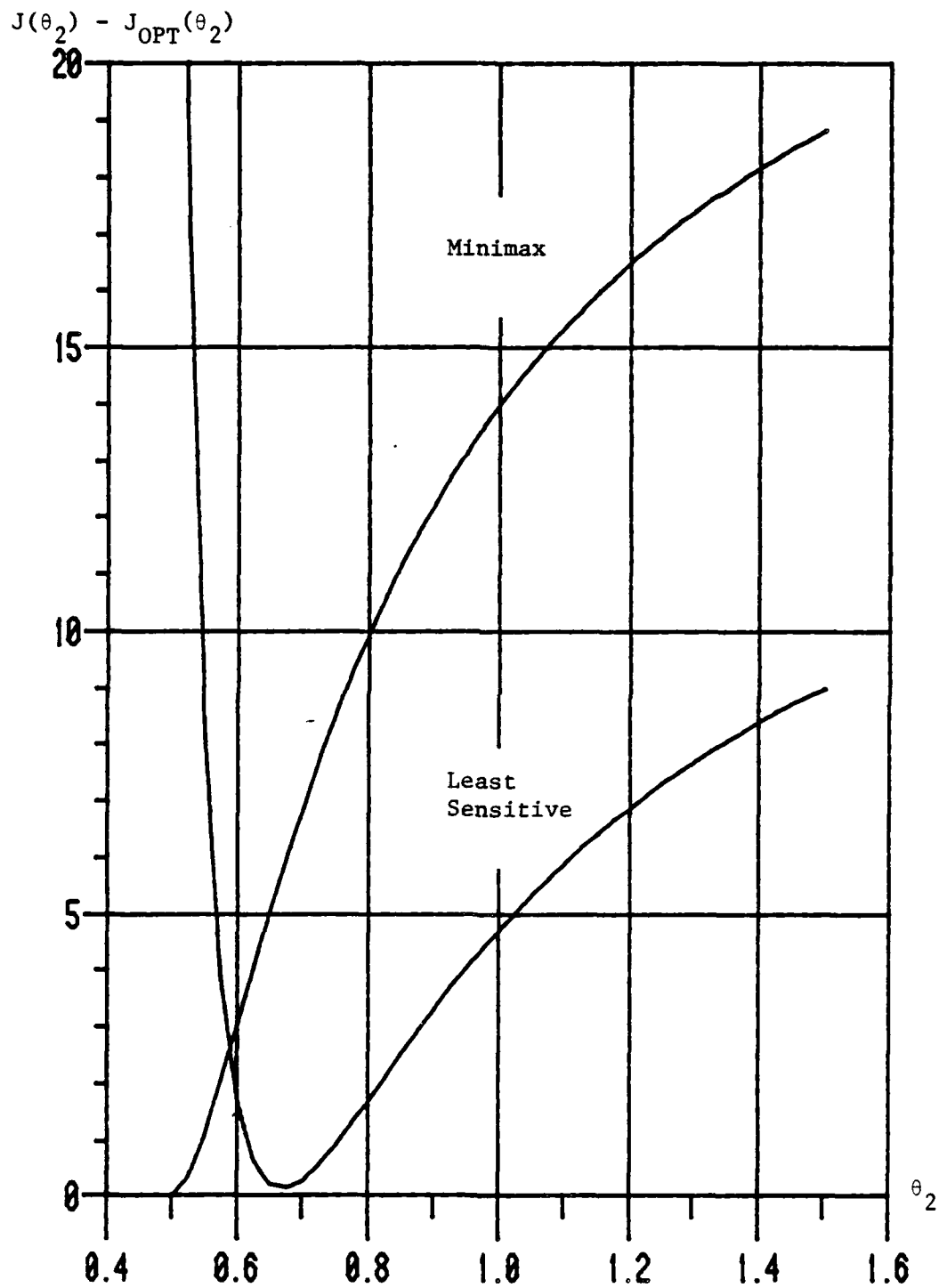


Figure 23.  $J_{\text{MM}} - J_{\text{OPT}}$  and  $J_{\text{LS}} - J_{\text{OPT}}$  vs.  $\theta_2$ ,  $\theta_1 = 1$ ,  $\theta_2 \in [0.5, 1.5]$ .



$\theta_G$	% difference
[0.9,1.1]	57.63%
[0.7,1.3]	3.39%
[0.5,1.5]	- 102.86%

Here the results are much more dramatic than in the previous example. For the set, [0.5,1.5], the maximum relative cost for the least sensitive controller is over 100% higher than the maximum relative cost of the minimax controller.

One cause for the difference in the behavior of the relative maxima of the two examples may be seen by referring back to the plots of  $J_{OPT}$  for the two examples given in Figures 4 and 11. As can be seen from these plots, the least sensitive controller must compensate for much higher worst-case costs when  $g_\theta = \theta_2$  is unknown. Also these worst-case costs increase more rapidly as the parameter set is widened for the  $g_\theta = \theta_2$  case.

This completes our observations for the two examples. These observations are useful when one has to make a choice between the two designs for a particular application. From Observations (2) and (3) of the two examples, one may detect a conflict in objectives. On the one hand, the least sensitive controllers exhibit superior performance over the wider range of parameter values. On the other hand, the least sensitive controllers can exhibit very poor performance over the worst-case values of parameters when the variance is substantial. If this worst-case performance is intolerable, one may prefer the minimax controllers over the least sensitive controllers. Therefore, either controller design may be preferable over the other, depending on the application.

#### 4.5. Conclusions

In conclusion, this section has dealt with the design of minimax controllers that minimize the worst-case cost. An equivalence between the solution of the minimax problem, (8), and the solution of the maximin problem, (9), has been sought. Problem (9) was shown to have a solution which could be found by setting the Fréchet differentials in Theorem 2 nonpositive. Theorem 2 also gave the requirements for the equivalence of the minimax problem, (8), and the maximin problem, (9). We noted that the method of proof of the equivalence of (8) and (9) when the uncertainties were in the noise covariance given by Looze, Poor, et al. [7] could not be used here since that proof relied on the creation of an error model that could not be formed in this case because the uncertainties were in the dynamics of the state model. Indeed we found that in the scalar case there were examples where (3) and (9) were not equivalent.

The scalar problem was then considered. The maximin solutions for a class of sets,  $\chi$ , was found by setting the Fréchet differentials of Theorem 2 nonpositive. We found the largest convex set in  $\mathbb{R}^3$  such that (8) and (9) are equivalent by examining the level surface of (99) at the maximin solution. This surface is a half-hyperboloid defined by (102). By examining the contours of (102) in the  $f_\theta - g_\theta$  plane, the  $f_\theta - h_\theta$  plane and the  $g_\theta - h_\theta$  plane at values of  $h_\theta = h$ ,  $g_\theta = g$  and  $f_\theta = f$ , respectively, we were able to find all the sets in  $\chi$  such that (8) and (9) were equivalent. This result was summarized in Theorem 3.

Finally the examples of Chapter 2 were considered, and Theorem 3 was used to show that the solutions to (8) and (9) were equivalent in these examples. The performance of the minimax controllers was compared with the performance of the least sensitive controllers of Chapter 2. It was seen that depending on the design objectives either controller may be preferable. Specifically, the least sensitive controllers were seen to exhibit superior performance over a wider range of parameter values, but the minimax controllers had lower maximum cost when the relative range of parameter values was substantial.

## CHAPTER 5

### A TWO-DIMENSIONAL SYSTEM

In this chapter, we consider the design of least sensitive and minimax controllers for a two-dimensional system with different uncertainties. These examples serve to present some important ideas about the design procedures and considerations for multivariable systems that were not illustrated in the previous examples.

The design of least sensitive controllers is considered first. The design procedure for these examples is the same as that of Chapter 3. However, with multivariable systems, the designer has the option of reducing the order of the controller dynamics, thus perhaps saving on some fixed costs. Therefore, the analysis of these examples will concentrate on the relative performance of the reduced-order controllers and the full-order controllers.

Next, the design of minimax controllers is considered. We saw in Chapter 4 that the design procedure for minimax controllers for multivariable systems in a general sense was not feasible, particularly in being able to show the equivalence of the minimax solution to the maximin solution through verification of the saddle-point condition of Theorem 1, Chapter 4. However, this chapter shows how the design may be carried out numerically for a specific example. First, the maximin controller can be found using a plot of the optimal cost for each value of the unknown parameter. Once the maximin solution is obtained, we can verify the saddle-point condition by investigating a plot of the cost associated with the use of the maximin controller for each value of the unknown parameter.

Finally performances of the minimax and the least sensitive controllers are compared under various circumstances, and the considerations one should make in choosing a particular design are discussed.

### 5.1. Formulation of Examples

We consider a two-dimensional system in canonical form with four possible unknown parameters:

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ \theta_3 \end{pmatrix} u dt + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix} \quad (157a)$$

$$dy = (\theta_4 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt + dv \quad (157b)$$

The nominal parameter vector is  $(\theta_1 \ \theta_2 \ \theta_3 \ \theta_4)' = (1 \ -1 \ 1 \ 1)'$ . We consider four examples, each with only one parameter unknown. In each example, we consider intervals with end points that are 10%, 30% and 50%, off the nominal value.

Example 5.1:  $\theta_1$  unknown,  $\theta_2 = -1$ ,  $\theta_3 = 1$ ,  $\theta_4 = 1$

Example 5.2:  $\theta_1 = 1$ ,  $\theta_2$  unknown,  $\theta_3 = 1$ ,  $\theta_4 = 1$

Example 5.3:  $\theta_1 = 1$ ,  $\theta_2 = -1$ ,  $\theta_3$  unknown,  $\theta_4 = 1$

Example 5.4:  $\theta_1 = 1$ ,  $\theta_2 = -1$ ,  $\theta_3 = 1$ ,  $\theta_4$  unknown

### 5.2. Least Sensitive Controllers

The gradient algorithm developed in Chapter 3 and given in the Appendix can be used to find first and second-order least sensitive controllers for these examples. However, an adequate starting point must be found in order to use the algorithm.

As a starting point for second-order controllers, we use the optimal controller for the nominal system in the block diagonal coordinates, (10). The nominal system is:

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u dt + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix} \quad (158a)$$

$$dy = (1 \ 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt + dv \quad (158b)$$

The optimal controller for this system is [3]:

$$u = - (0 \ 1) P \hat{x} \quad (159)$$

where

$$d\hat{x} = \left[ \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P - \Sigma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \hat{x} dt + \Sigma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} dy \quad (160)$$

and  $P = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}$  satisfies:

$$\begin{aligned} & \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} + \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & - \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \ 1) \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} = 0 \end{aligned} \quad (161)$$

and  $\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{pmatrix}$  satisfies:

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{pmatrix} + \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ - \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0) \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{pmatrix} = 0 \quad . \quad (162)$$

Equation (161) reduces to three equations:

$$2P_2 + 1 - P_2^2 = 0 \quad (163a)$$

$$P_3 + P_1 - P_2 - P_2 P_3 = 0 \quad (163b)$$

$$2P_2 - 2P_3 + 1 - P_3^2 = 0 \quad . \quad (163c)$$

The positive definite solution to Equations (163) is

$$P = \begin{pmatrix} 4.6954 & 2.4142 \\ 2.4142 & 1.6131 \end{pmatrix} \quad . \quad (164)$$

Equation (162) reduces to three equations:

$$2\Sigma_2 + 1 - \Sigma_1^2 = 0 \quad (165a)$$

$$\Sigma_3 + \Sigma_1 - \Sigma_2 - \Sigma_1 \Sigma_2 = 0 \quad (165b)$$

$$2\Sigma_2 - 2\Sigma_3 + 1 - \Sigma_2^2 = 0 \quad . \quad (165c)$$

The positive definite solution to Equations (165) is

$$\Sigma = \begin{pmatrix} 1.732 & 1 \\ 1 & 1 \end{pmatrix} \quad . \quad (166)$$

With  $P$  and  $\Sigma$  given in (164) and (166), Equations (159) and (160) reduce to

$$u = - (2.4142 \quad 1.6131)\hat{x} \quad (167)$$

$$d\hat{x} = \begin{pmatrix} -1.732 & 1 \\ -2.4142 & -2.6131 \end{pmatrix} \hat{x} dt + \begin{pmatrix} 1.7321 \\ 1 \end{pmatrix} dy \quad (168)$$

In the block diagonal coordinates, (10), the optimal controller is of the form:

$$u = - (0 \quad 1)z \quad (169)$$

$$dz = \begin{pmatrix} 0 & 1 \\ p_1 & p_2 \end{pmatrix} z dt + \begin{pmatrix} p_3 \\ p_4 \end{pmatrix} dy \quad (170)$$

We match the  $Y(s)$  to  $U(s)$  transfer functions of these two controllers. For controllers represented by (167) and (168),

$$\begin{aligned} \frac{U(s)}{Y(s)} &= - (2.4142 \quad 1.6131) \begin{pmatrix} s + 1.7321 & -1 \\ 2.4142 & s + 2.6131 \end{pmatrix}^{-1} \begin{pmatrix} 1.7321 \\ 1 \end{pmatrix} s \\ &\quad - (2.4142 \quad 1.6131) \begin{pmatrix} s + 2.6131 & 1 \\ -2.4142 & s + 1.7321 \end{pmatrix} \begin{pmatrix} 1.7321 \\ 1 \end{pmatrix} s \\ &= \frac{- (5.7946 s + 9.3897)s}{s^2 + 4.3452 s + 6.9403} \quad (171) \end{aligned}$$

For controllers represented by (169) and (170),



$$\begin{aligned}
\frac{U(s)}{Y(s)} &= (0 \ 1) \begin{pmatrix} s & -1 \\ -p_1 & s-p_2 \end{pmatrix}^{-1} \begin{pmatrix} p_3 \\ p_4 \end{pmatrix} s \\
&= (0 \ 1) \begin{pmatrix} s-p_2 & 1 \\ p_1 & s \end{pmatrix} \begin{pmatrix} p_3 \\ p_4 \end{pmatrix} s \\
&\quad \frac{1}{s^2 - p_2 s - p_1} \\
\frac{U(s)}{Y(s)} &= \frac{(p_4 s + p_1 p_3) s}{s^2 - p_2 s - p_1} \quad (172)
\end{aligned}$$

Comparison of (171) and (172) gives the desired starting point for the gradient algorithm:

$$\begin{aligned}
p_1 &= -6.9403 \\
p_2 &= -4.3452 \\
p_3 &= 1.3529 \\
p_4 &= -5.7947 \quad (173)
\end{aligned}$$

To find a starting point for first-order controllers, we find the  $Y(s)$  to  $U(s)$  transfer function for the first-order controller and attempt to match this transfer function to (172). The first-order controller is

$$u = z \quad (174)$$

$$dz = \bar{p}_1 z dt + \bar{p}_2 dy \quad (175)$$

Then,

$$\frac{U(s)}{Y(s)} = \frac{\bar{p}_2 s}{s - \bar{p}_1} \quad (176)$$

To facilitate a comparison between (176) and (172), we attempt to cancel the zero in (172)

$$\frac{U(s)}{Y(s)} = \frac{p_4 s \left( s + \frac{p_1 p_3}{p_4} \right)}{s^2 - p_2 s - p_1}$$

$$\frac{U(s)}{Y(s)} = \frac{p_4 s}{s - \left( p_2 + \frac{p_1 p_3}{p_4} \right) - \frac{\left[ p_1 - \frac{p_1 p_3}{p_4} \left( p_2 + \frac{p_1 p_3}{p_4} \right) \right]}{s + \frac{p_1 p_3}{p_4}}} \quad (177)$$

We compare (176) to (177) and set

$$\bar{p}_2 = p_4 = -4.1743$$

$$\bar{p}_1 = p_2 + \frac{p_1 p_3}{p_4} = -5.7947 \quad (178)$$

where we have neglected the remainder in the denominator of Equation (177). However, this remainder is not negligible, so we seek an improvement in this starting point. We can use the gradient algorithm to make this improvement by noting that choosing a degenerate single point distribution corresponds to the case where the parameter value is known. Therefore, by using (178) as a starting point for the gradient algorithm applied to the nominal system, a more exact starting point is found. This point is

$$\bar{p}_1 = -5.9235$$

$$\bar{p}_2 = -8.3097 \quad (179)$$

Note that these are the control parameters corresponding to the optimal first-order controller for the nominal system. We use these parameters as the starting point for determining first-order controllers using the gradient algorithm.

We now consider the design and analysis of least sensitive controllers for Examples 5.1 through 5.4, for various intervals spread evenly about the nominal values. As in Chapter 3, uniform distributions are induced on these intervals. The same type of approximating distributions is found using Equations (34), (35a), (39) and (41) of Chapter 3, choosing  $N=3$ . We can use this low choice of  $N$  since the corresponding controllers have similar performances as controllers designed for  $N$  large have. Therefore, we can save on computations without losing much information pertinent to our analysis.

Again we choose as a performance measure a plot of the difference between the cost,  $J_{LS}(\theta)$ , incurred by the least sensitive controller, and the optimal cost,  $J_{OPT}(\theta)$ , for each value of the parameter over the range of uncertainty. This is a plot of the additional cost incurred by the least sensitive controller over the minimum cost if the parameter were known.

Now consider Example 5.1:

$$\theta_1 \text{ unknown, } \theta_2 = -1, \theta_3 = 1, \theta_4 = 1$$

with the uniform distribution induced on the 10%, 30% and 50% spreads:

$$\theta_1 \sim U[0.9, 1.1]$$

$$\theta_1 \sim U[0.7, 1.3]$$

$$\theta_1 \sim U[0.5, 1.5]$$

(180)

The gradient algorithm is used to calculate first and second-order least sensitive controllers for these distributions, along with the corresponding average cost. The results are:

Distribution	Controller Order	$P_1$	$P_2$	$P_3$	$P_4$	$J_{AVG}$
U[0.9,1.1]	1	-5.9266	-8.4104			28.6159
	2	-7.2769	-4.3981	1.3724	-5.8667	27.2015
U[0.7,1.3]	1	-5.9822	-9.100			31.4066
	2	-9.1375	-4.7185	1.4852	-6.3612	30.0375
U[0.5,1.5]	1	-6.1337	-10.1914			36.1568
	2	-11.3105	-5.1174	1.6304	-7.1322	34.7152

The plots of  $J_{LS}(\theta) - J_{OPT}(\theta)$  for the first and second-order controllers are shown in Figures 24, 25 and 26 for each distribution. Several observations can be made from these curves.

- 1) The performance of the second-order controllers exhibits most of the properties observed in Chapter 3, which are related to the basic shape of the performance curve. Since the first-order controllers have the same basic shape, they also exhibit these properties. These properties include the fact that less performance is sacrificed at near nominal values than at off nominal values, and that the controllers are biased towards parameter values associated with higher costs.
- 2) The savings in using a second-order controller over a first-order controller relative to the total average cost of the first-order controller is significant, but relatively low. We can quantify this by looking at the percent decrease in the average cost of the second-order controllers relative to the cost of the first-order controllers.

Distribution	Decrease in $J_{AVG}$
U[0.9,1.1]	5.2%
U[0.7,1.3]	4.6%
U[0.5,1.5]	4.2%

- 3) Since we are obligated to incur at least the optimal cost of each value of the parameter, there is a maximum amount one can save over the average cost of

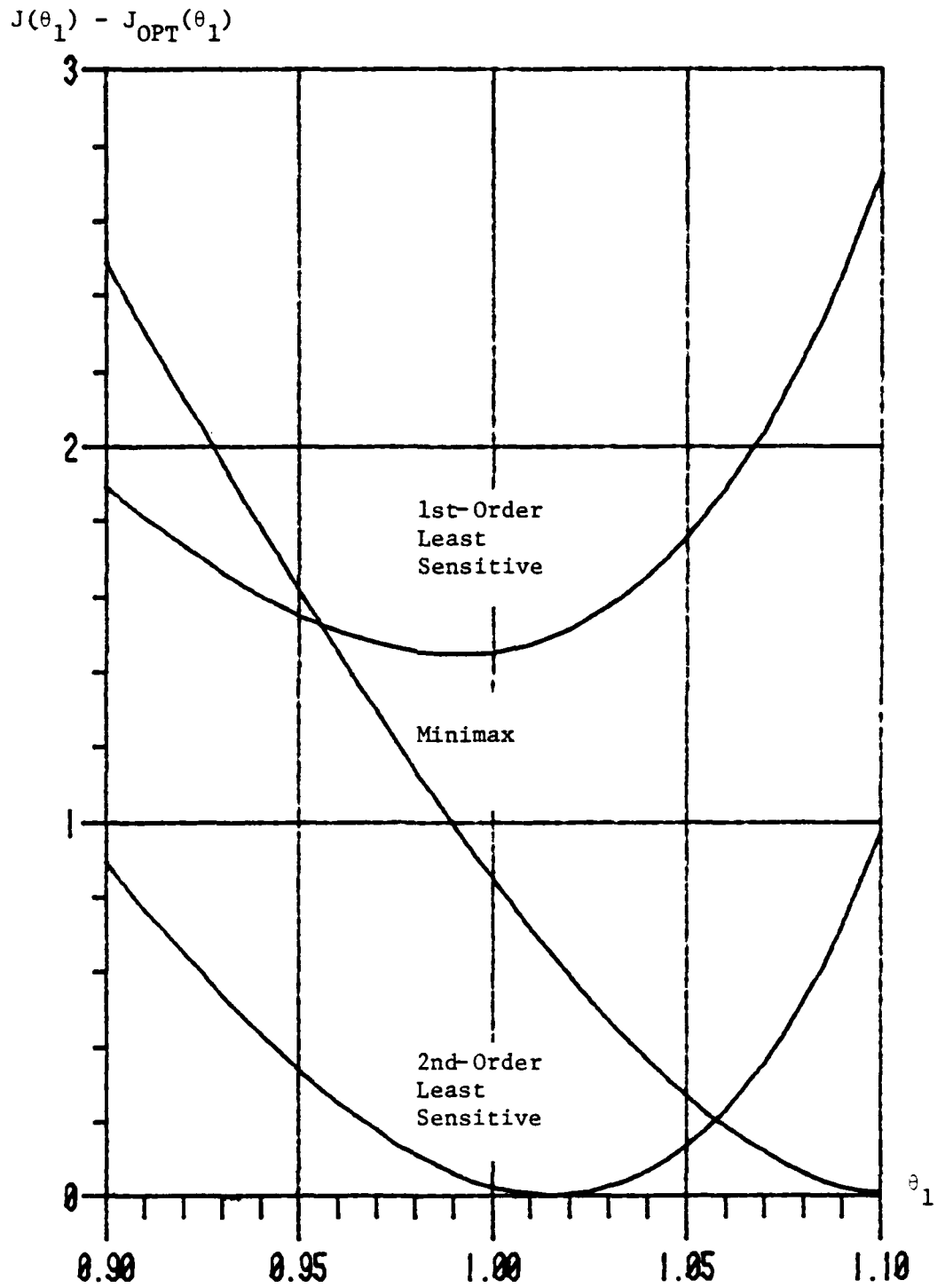


Figure 24.  $J_{LS} - J_{OPT}$  vs.  $\theta_1$  for 1st and 2nd-order controllers,  $\theta_1 \sim U[0.9, 1.1]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_1$ ,  $\theta_1 \in [0.9, 1.1]$ .

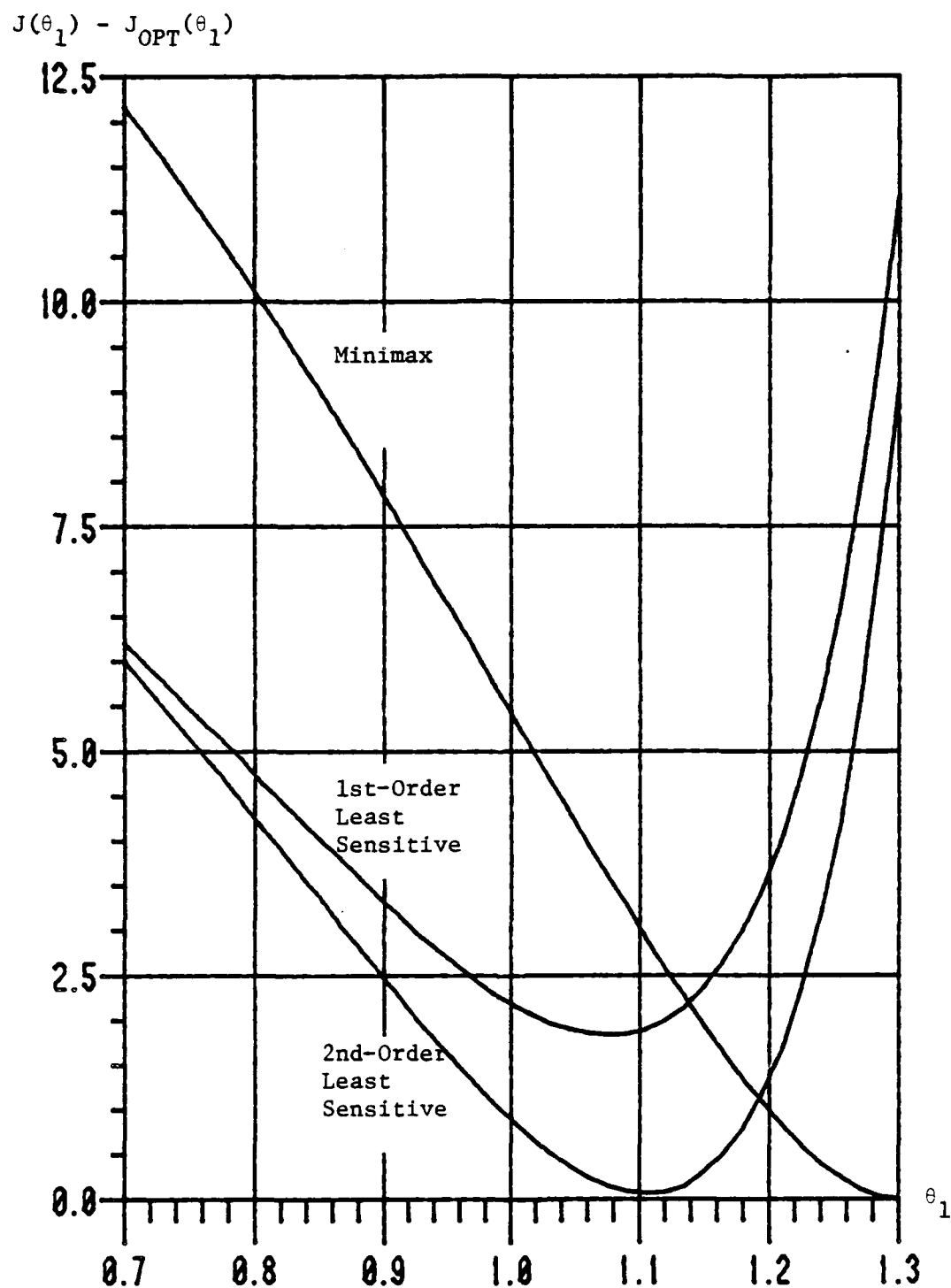


Figure 25.  $J_{LS} - J_{OPT}$  vs.  $\theta_1$  for 1st and 2nd-order controllers,  $\theta_1 \sim U[0.7, 1.3]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_1$ ,  $\theta_1 \in [0.7, 1.3]$ .

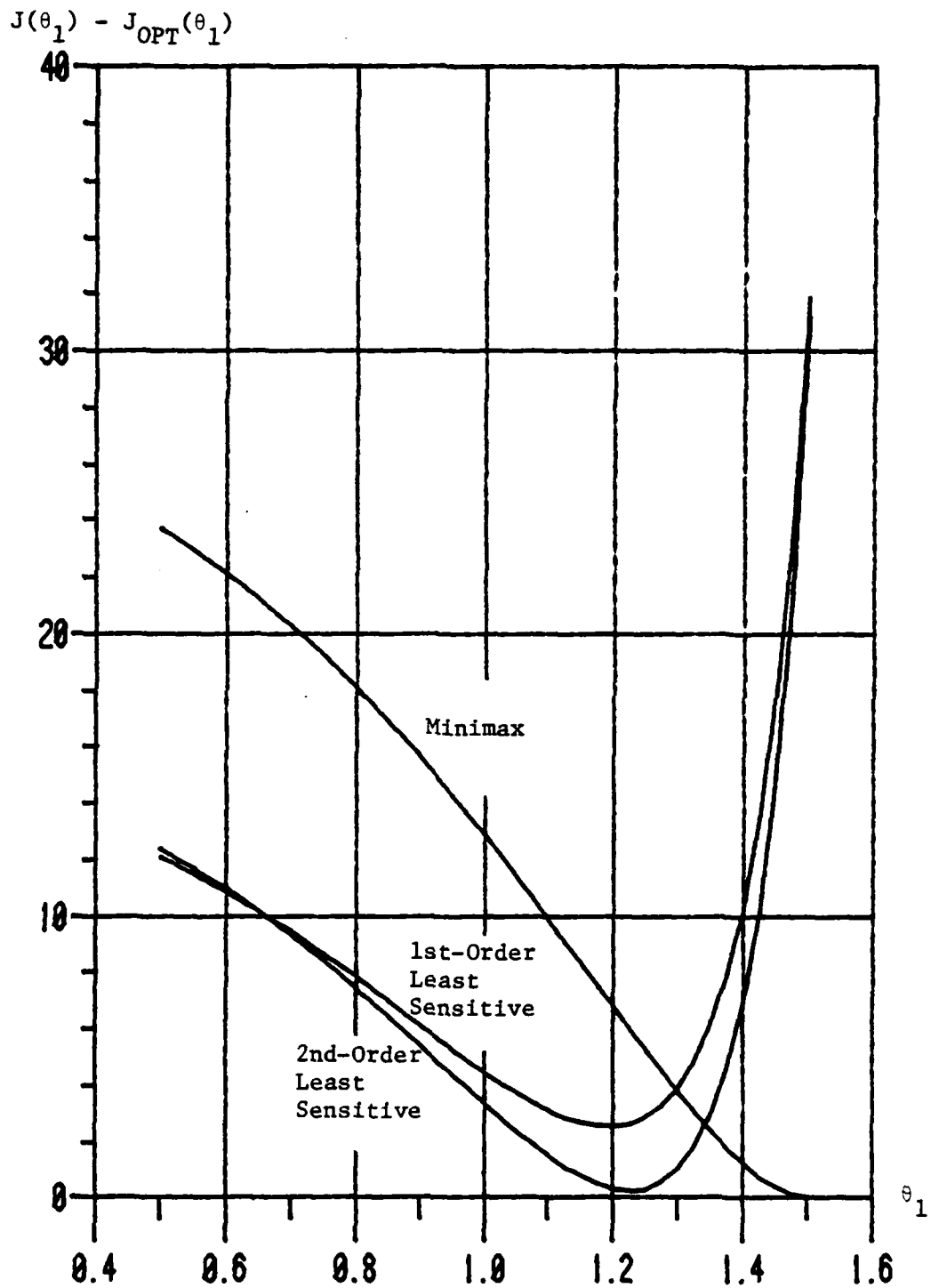


Figure 26.  $J_{LS} - J_{OPT}$  vs.  $\theta_1$  for 1st and 2nd-order controllers,  $\theta_1 \sim U[0.5, 1.5]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_1$ ,  $\theta_1 \in [0.5, 1.5]$ .

the first-order controllers. Figures 24, 25 and 26 show that large percentages of the maximum possible savings are realized by the second-order controllers. For each distribution, we find the percentage of the maximum possible savings over the first-order costs that the second-order controllers realize.

Distribution	Percent of Maximum Savings
U[0.9,1.1]	81.9%
U[0.7,1.3]	34.6%
U[0.5,1.5]	21.1%

Note that this percentage decreases as the uncertainty increases so that the two controllers are more comparable as less sensitivity is desired.

4) From Figures 24, 25 and 26 we can see that the difference between the first and second-order controllers is greater for near nominal values of the parameter and less for off nominal values. Therefore, more performance is sacrificed at near nominal values than at off nominal values when the first-order controllers are used instead of second-order controllers.

These observations show that the first-order controllers have fairly good performance relative to the total cost incurred by each type of controller, but that the second-order controllers can realize a large amount of the total possible improvement in performance. Also, the first and second-order controllers become more comparable as the range of uncertainty increases. We will discuss what these observations mean when one is considering a specific design after the remaining examples are analyzed.

Before proceeding to the next example, note that Observations (1) and (4) above will generally hold for the remaining examples. Therefore, we will concentrate on observations similar to (2) and (3) in the discussions of the remaining examples.



Next, consider Example 5.2:

$$\theta_1 = 1, \theta_2 \text{ unknown}, \theta_3 = 1, \theta_4 = 1$$

with the uniform distribution indicated on 10%, 30% and 50% spreads:

$$\begin{aligned} \theta_2 &\sim U[-1.1, -0.9] \\ \theta_2 &\sim U[-1.3, -0.7] \\ \theta_2 &\sim U[-1.5, -0.5] \end{aligned} \quad (181)$$

The gradient algorithm is used to calculate first and second-order least sensitive controllers for these distributions, along with the corresponding average cost. The results are:

Distribution	Controller Order	$p_1$	$p_2$	$p_3$	$p_4$	$J_{AVG}$
U[-1.1, -0.9]	1	-5.9628	-8.3610			28.3333
	2	-6.9373	-4.3413	1.3521	-5.8081	26.8616
U[-1.3, -0.7]	1	-6.2947	-8.7943			29.2138
	2	-6.9181	-4.3118	1.3452	-5.9166	27.4057
U[-1.5, -0.5]	1	-7.0611	-9.7934			31.2339
	2	-6.9046	-4.2621	1.3048	-6.1376	28.5435

The plots of  $J_{LS}(\theta) - J_{OPT}(\theta)$  for first and second-order controllers are shown in Figures 27, 28 and 29 for each distribution. Two similar observations to those of Example 5.1 will be made here.

- 1) We again calculate the decrease in average cost of the second-order controllers relative to the average cost of the first-order controllers.

Distribution	Decrease in $J_{AVG}$
U[-1.1, -0.9]	5.2%
U[-1.3, -0.7]	8.1%
U[-1.5, -0.5]	8.6%

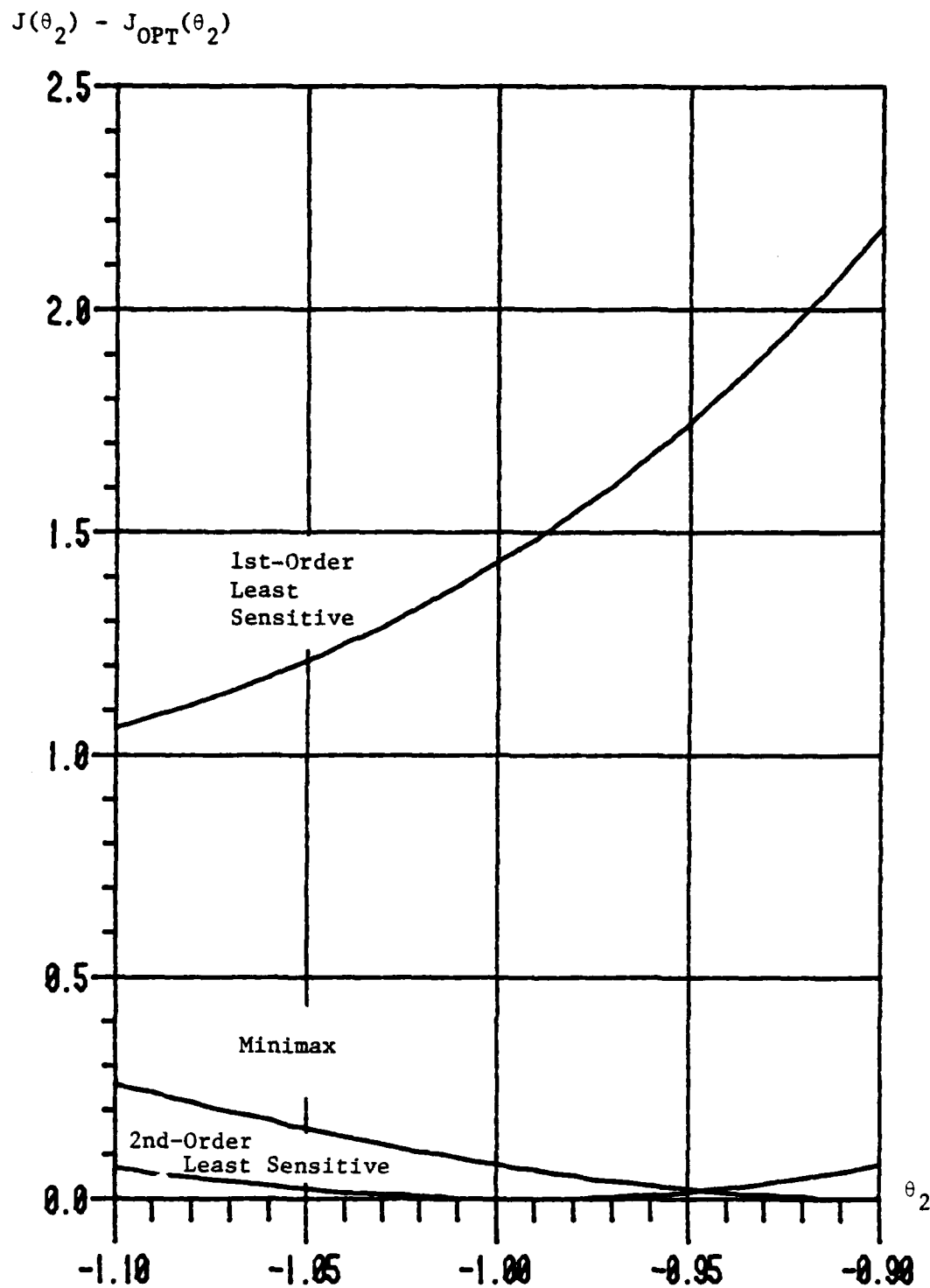


Figure 27.  $J_{LS} - J_{OPT}$  vs.  $\theta_2$  for 1st and 2nd-order controllers,  
 $\theta_2 \sim U[-1.1, -0.9]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_2$ ,  $\theta_2 \in [-1.1, -0.9]$ .

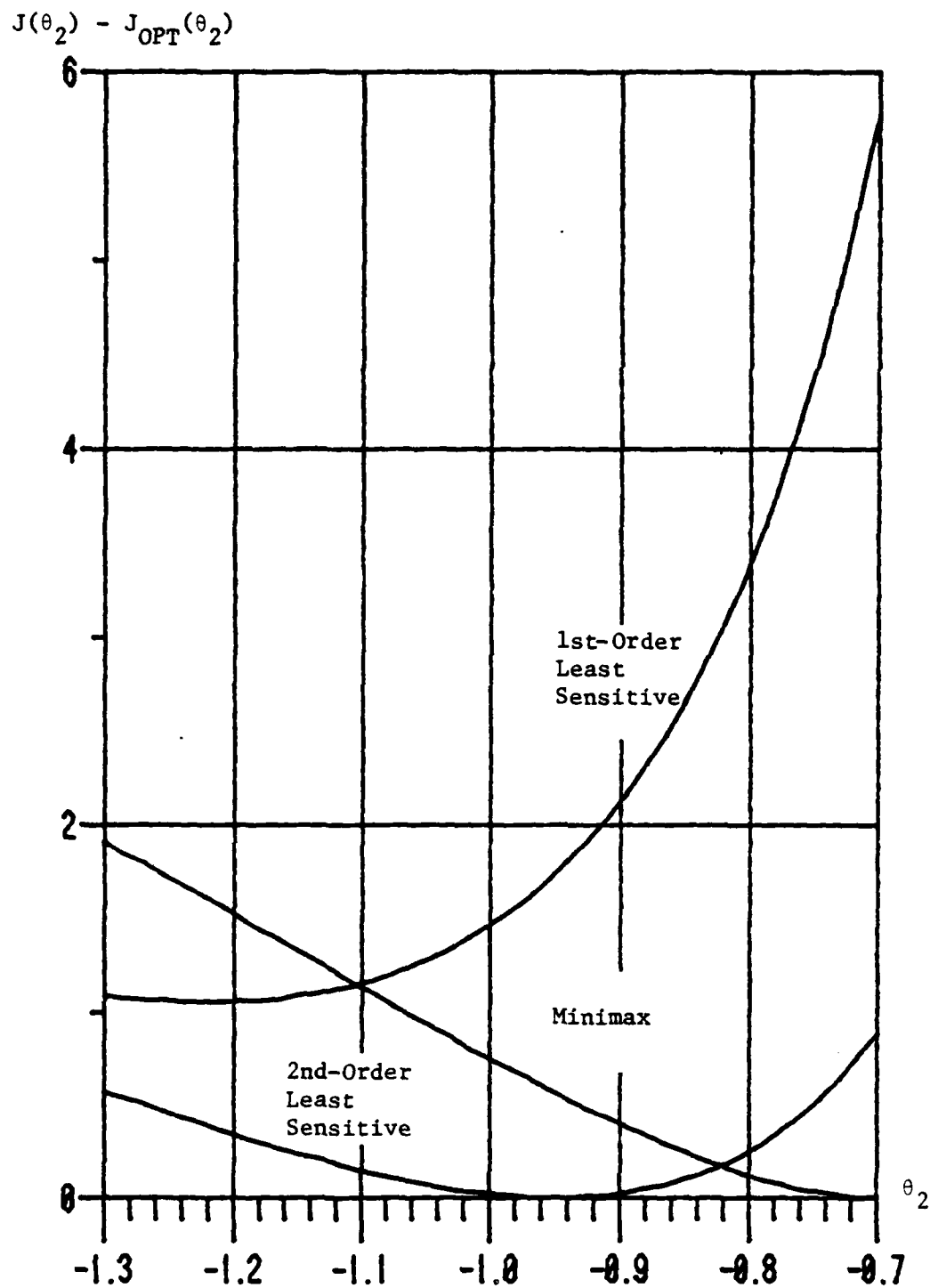


Figure 28.  $J_{LS} - J_{OPT}$  vs.  $\theta_2$  for 1st and 2nd-order controllers,  $\theta_2 \in U[-1.3, -0.7]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_2$ ,  $\theta_2 \in [-1.3, -0.7]$ .

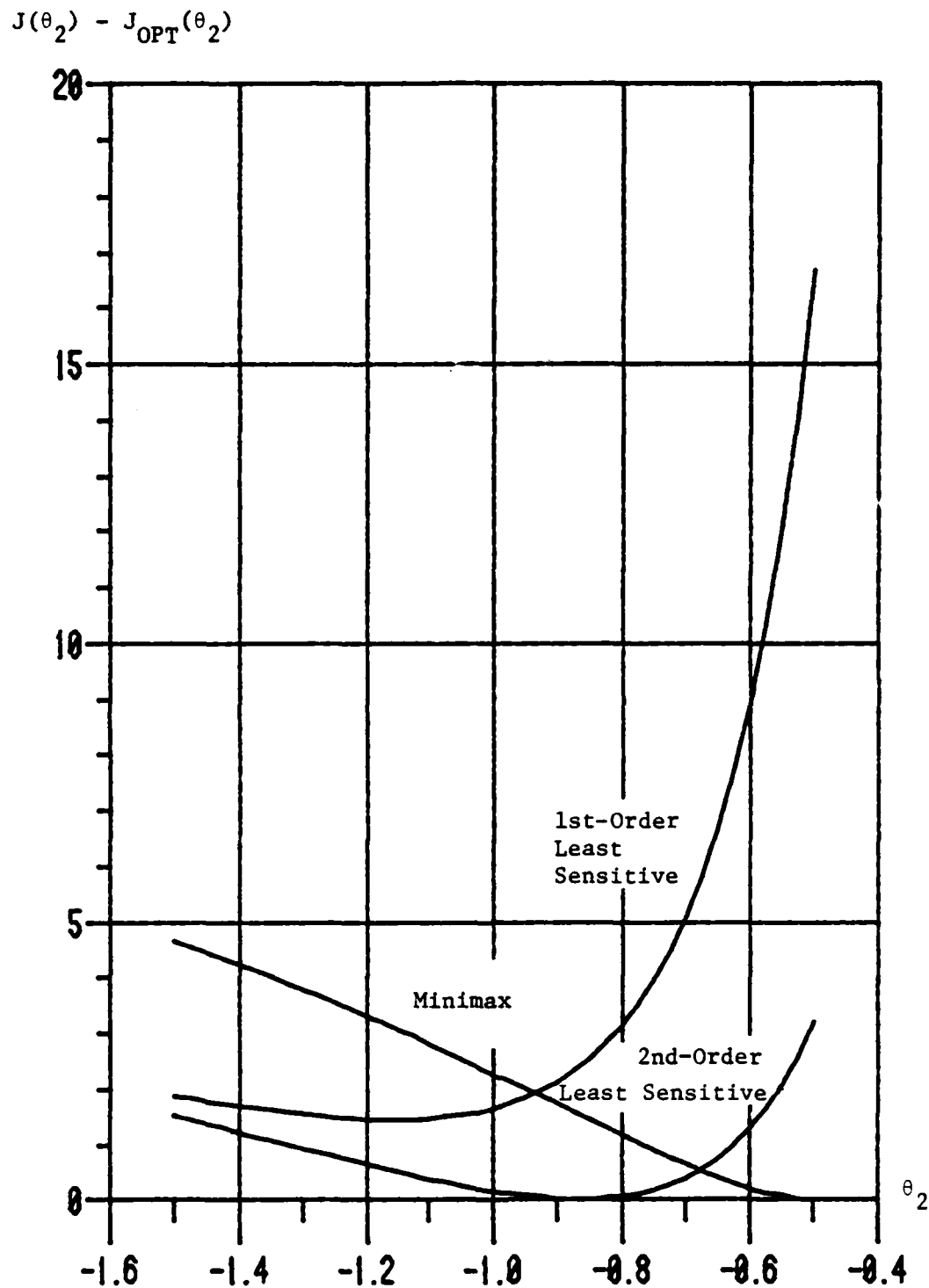


Figure 29.  $J_{LS} - J_{OPT}$  vs.  $\theta_2$  for 1st and 2nd-order controllers,  
 $\theta_2 \sim U[-1.5, -0.5]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_2$ ,  $\theta_2 \in [-1.5, -0.5]$ .

Note that as in the previous example, the savings are significant, but relatively low.

2) As in the previous example we calculate the percentage of the maximum savings over the first-order controller that the second-order controller realizes.

Distribution	Percentage of Maximum Savings
$U[-1.1, -0.9]$	98.3%
$U[-1.3, -0.7]$	88.7%
$U[-1.5, -0.5]$	80.4%

Note that these results are much more dramatic than those in Example 5.1, with only a slight decrease in savings as more parameter insensitivity is desired.

Thus our observations in this example are similar to Example 5.1 except that the improvement of second-order controllers in this example is generally greater. This greater improvement may be a result of the fact that control effort for  $\theta_2$  has more effect on the stable subspace of the states and thus the performance is more responsive to the greater control effort of the second-order controllers.

Next, we consider Example 5.3:

$$\theta_1 = 1, \theta_2 = -1, \theta_3 \text{ unknown}, \theta_4 = 1$$

with the uniform distribution induced on 10%, 30% and 50% spreads:

$$\theta_3 \sim U[0.9, 1.1]$$

$$\theta_3 \sim U[0.7, 1.3]$$

$$\theta_3 \sim U[0.5, 1.5]$$

(182)

The gradient algorithm is used to calculate first and second-order least sensitive controllers for these distributions, along with the corresponding average cost. The results are:

Distribution	Controller Order	$P_1$	$P_2$	$P_3$	$P_4$	$J_{AVG}$
U[0.9,1.1]	1	-5.9157	-8.4212			28.7810
	2	-7.5486	-4.4366	1.3792	-5.8814	22.4062
U[0.7,1.3]	1	-5.9351	-9.3592			33.5228
	2	-12.2170	-5.2161	1.5533	-6.6179	32.4224
U[0.5,1.5]	1	-6.2351	-11.6023			45.3055
	2	-22.2779	-6.9242	1.8494	-8.3323	44.3532

The plots of  $J_{LS}(\theta) - J_{OPT}(\theta)$  for first and second-order controllers are shown in Figures 30, 31 and 32 for each distribution. Again we make the same types of observations.

- 1) We calculate the decrease in average cost of the second-order controllers relative to the average cost of the first-order controllers.

Distribution	Decrease in $J_{AVG}$
U[0.9,1.1]	4.8%
U[0.7,1.3]	3.3%
U[0.5,1.5]	2.1%

Again, the savings is significant, but relatively low.

- 2) We calculate the percentage of the maximum possible savings over the first-order controllers that the second-order controllers realize.

Distribution	Percent of Maximum Savings
U[0.9,1.1]	75.9%
U[0.7,1.3]	18.1%
U[0.5,1.5]	3.3%

Note that as the variance increases the relative savings that the second-order controller realizes drops to almost nothing. Indeed, we can see from Figure 32

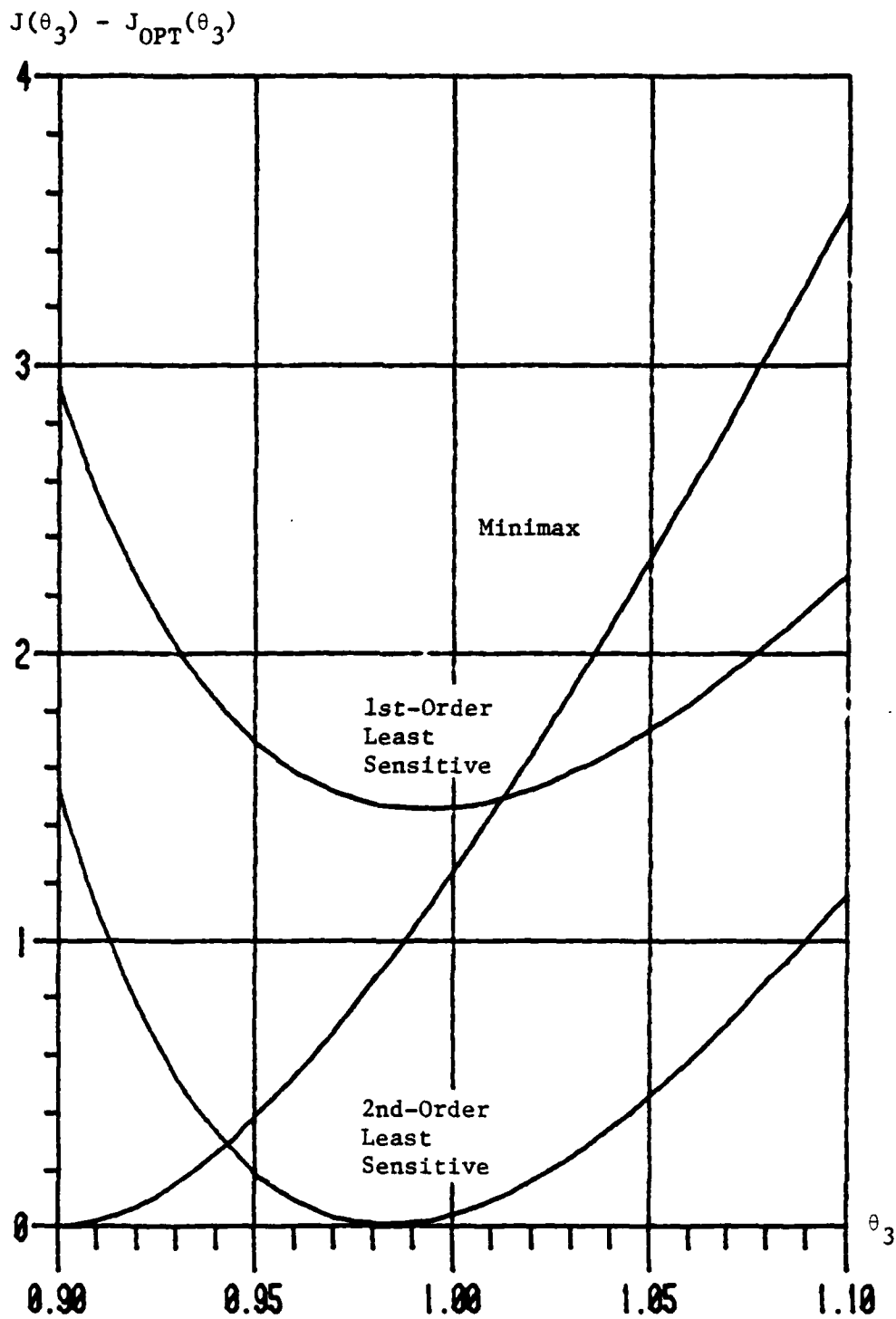


Figure 30.  $J_{LS} - J_{OPT}$  vs.  $\theta_3$  for 1st and 2nd-order controllers,  $\theta_3 \sim U[0.9, 1.1]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_3$ ,  $\theta_3 \in [0.9, 1.1]$ .

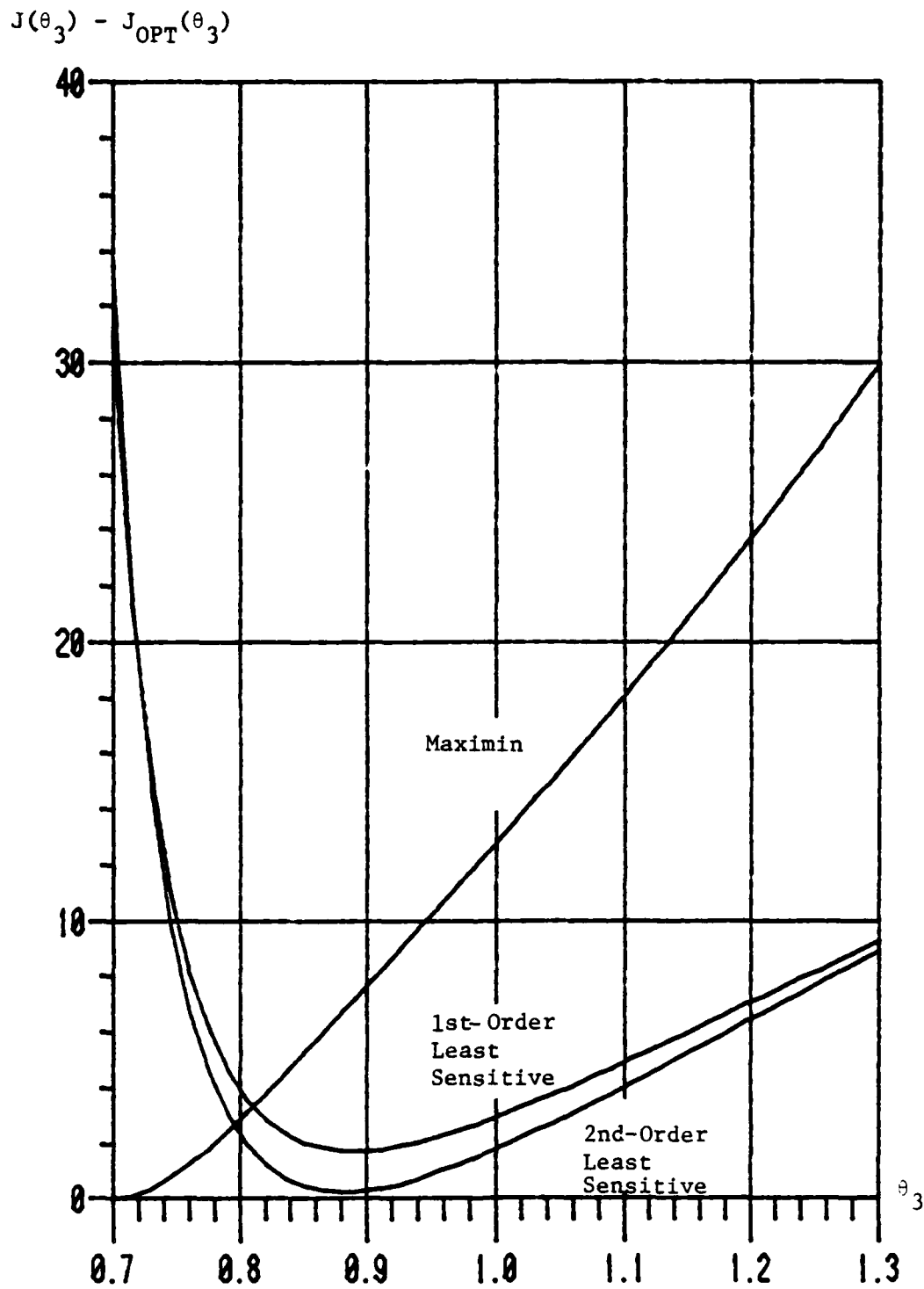


Figure 31.  $J_{LS} - J_{OPT}$  vs.  $\theta_3$  for 1st and 2nd-order controllers,  $\theta_3 \sim U[0.7, 1.3]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_3$ ,  $\theta_3 \in [0.7, 1.3]$ .



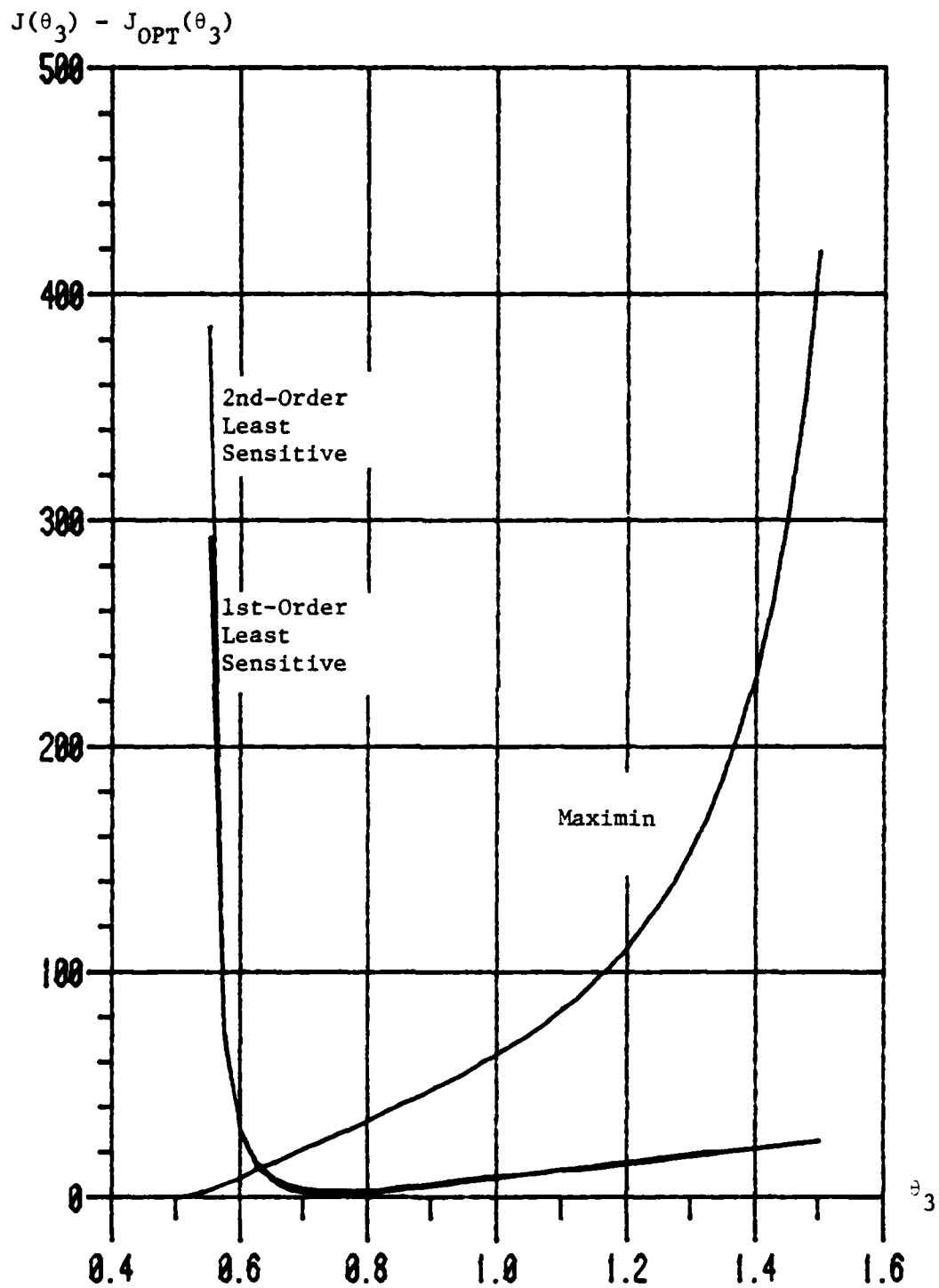


Figure 32.  $J_{LS} - J_{OPT}$  vs.  $\theta_3$  for 1st and 2nd-order controllers,  $\theta_3 \sim U[0.5, 1.5]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_3$ ,  $\theta_3 \in [0.5, 1.5]$ .

that the performances of the first and second-order controllers are almost identical for the  $U[0.5,1.5]$  distribution.

We will see that the results for this example are similar to those of Example 5.4. We now consider Example 5.4:

$$\theta_1 = 1, \theta_2 = -1, \theta_3 = 1, \theta_4 \text{ unknown}$$

with the uniform distribution induced on 10%, 30% and 50% spreads:

$$\begin{aligned} \theta_4 &\sim U[0.9,1.1] \\ \theta_4 &\sim U[0.7,1.3] \\ \theta_4 &\sim U[0.5,1.5] \end{aligned} \quad (183)$$

The gradient algorithm is used to calculate first and second-order least sensitive controllers for these distributions, along with the corresponding average cost. The results are:

Distribution	Controller Order	$p_1$	$p_2$	$p_3$	$p_4$	$J_{AVG}$
$U[0.9,1.1]$	1	-5.9154	-8.4129			28.7238
	2	-7.5431	-4.4347	1.3778	-5.8746	27.3502
$U[0.7,1.3]$	1	-5.9273	-9.2932			33.0063
	2	-12.2456	-5.2146	1.5449	-6.5604	31.9225
$U[0.5,1.5]$	1	-6.2092	-11.4348			43.7246
	2	-22.4724	-6.9572	1.8310	-8.2123	42.8125

The plots of  $J_{LS}(\theta) - J_{OPT}(\theta)$  for first and second-order controllers are shown in Figures 33, 34 and 35 for each distribution.

Again, we make the same types of observations.

- 1) We calculate the decrease in average cost of the second-order controllers relative to the average cost of the first-order controllers.

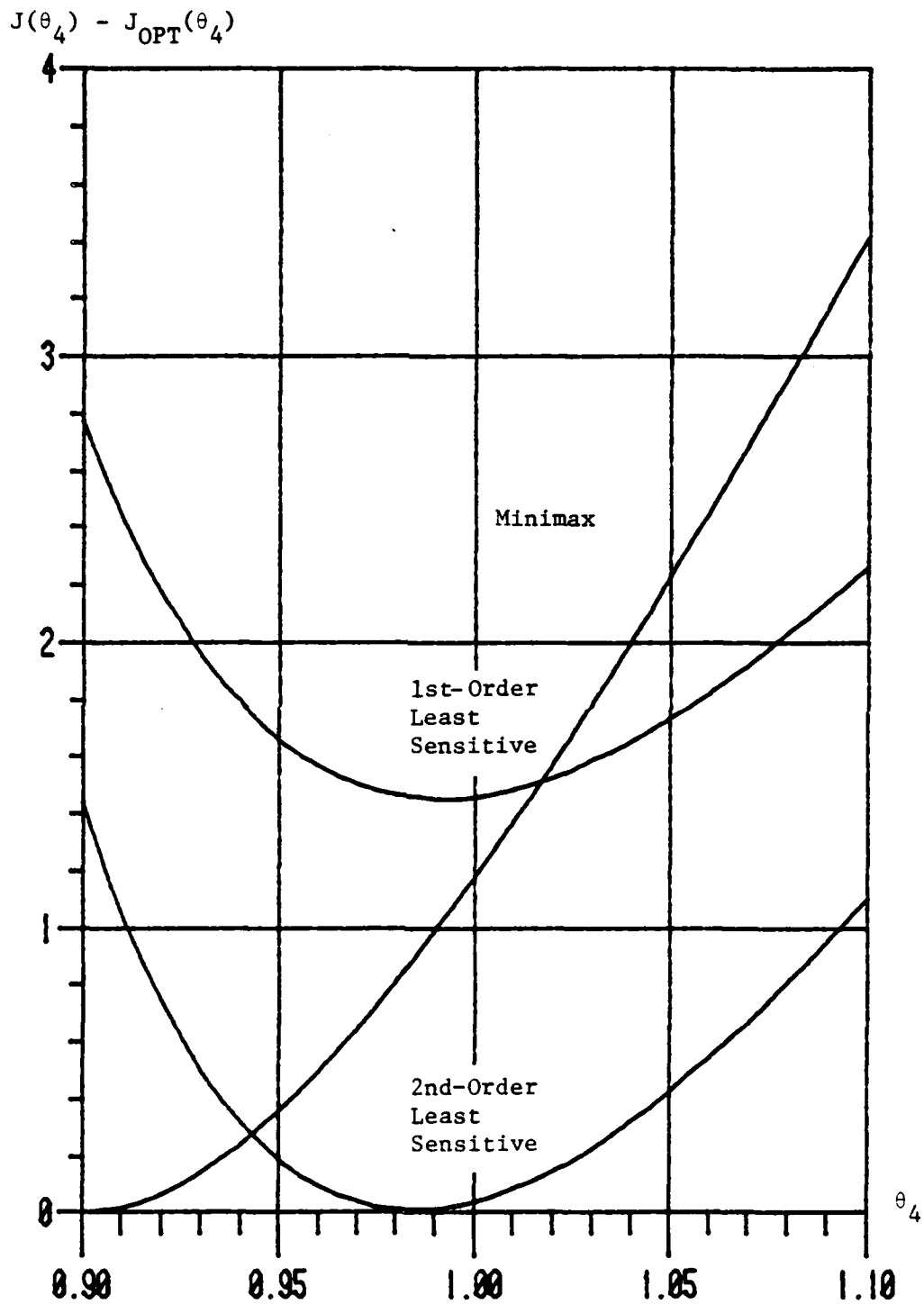


Figure 33.  $J_{\text{LS}} - J_{\text{OPT}}$  vs.  $\theta_4$  for 1st and 2nd-order controllers,  $\theta_4 \sim U[0.9, 1.1]$ ; and  $J_{\text{MM}} - J_{\text{OPT}}$  vs.  $\theta_4$ ,  $\theta_4 \in [0.9, 1.1]$ .

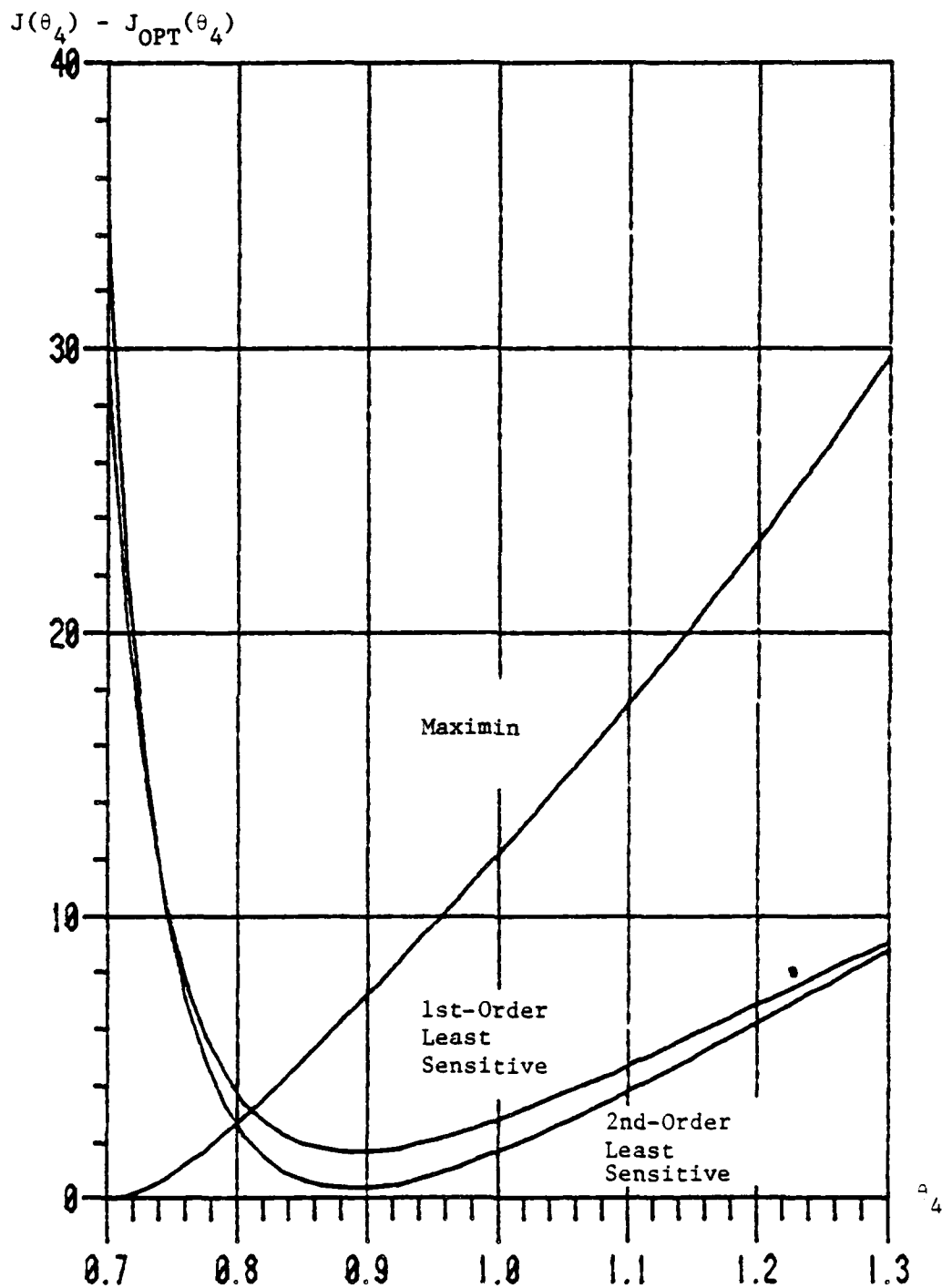


Figure 34.  $J_{\text{LS}} - J_{\text{OPT}}$  vs.  $\theta_4$  for 1st and 2nd-order controllers,  $\theta_4 \in U[0.7, 1.3]$ ; and  $J_{\text{MM}} - J_{\text{OPT}}$  vs.  $\theta_4$ ,  $\theta_4 \in [0.7, 1.3]$ .

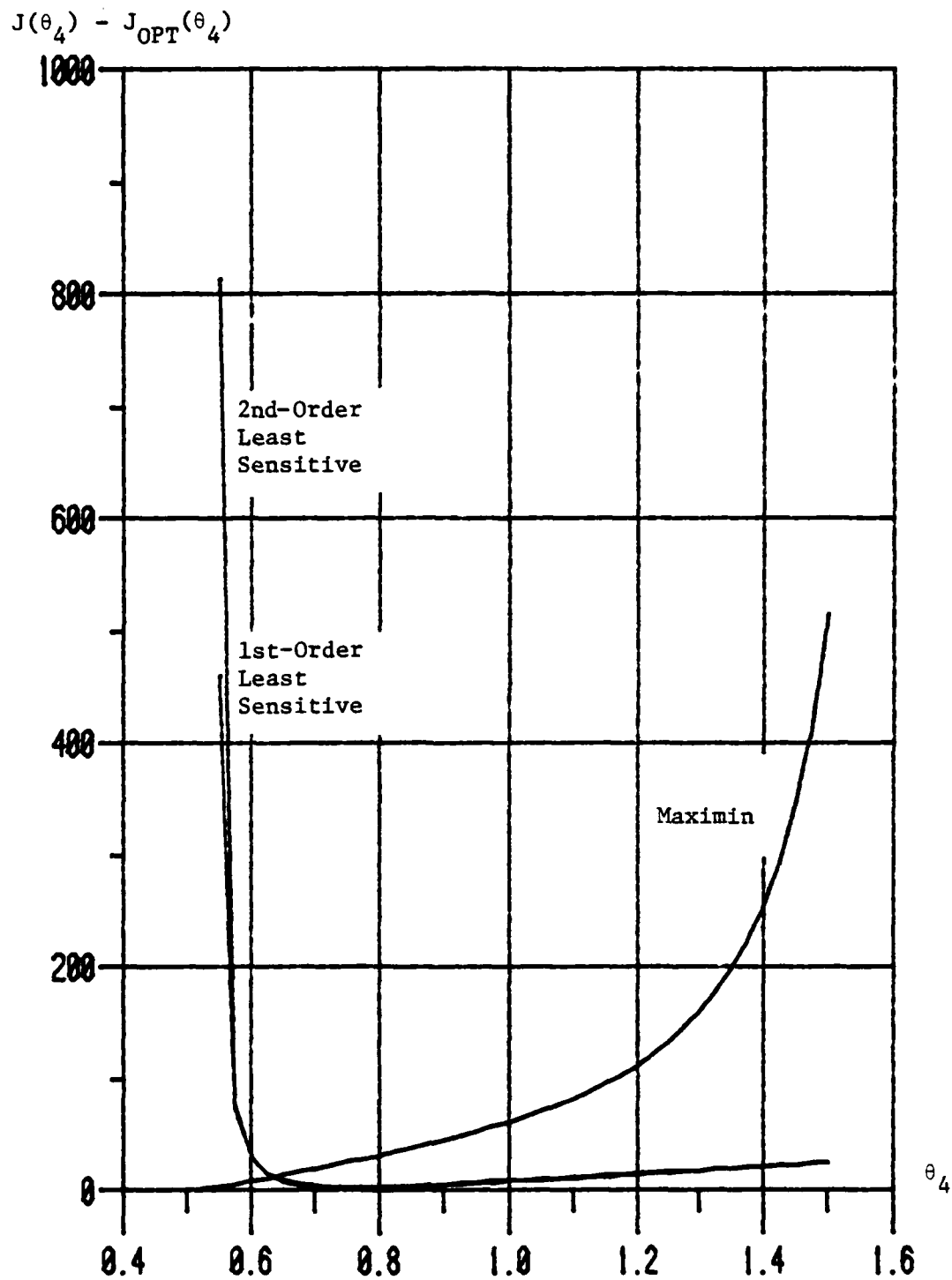


Figure 35.  $J_{LS} - J_{OPT}$  vs.  $\theta_4$  for 1st and 2nd-order controllers,  $\theta_4 \sim U[0.5, 1.5]$ ; and  $J_{MM} - J_{OPT}$  vs.  $\theta_4$ ,  $\theta_4 \in [0.5, 1.5]$ .

Distribution	Decrease in $J_{AVG}$
U[0.9,1.1]	4.8%
U[0.7,1.3]	3.3%
U[0.5,1.5]	2.1%

Again, the savings is significant, but relatively low.

2) We calculate the percentage of the maximum possible savings over the first-order controllers that the second-order controllers realize.

Distribution	Percent of Maximum Savings
U[0.9,1.1]	76.8%
U[0.7,1.3]	15.2%
U[0.5,1.5]	4.2%

Note that as in Example 5.3 the relative savings realized by the second-order controller drops to almost nothing, and we can see from Figure 35 that the performances of the first and second-order controllers are almost identical.

Examples 5.3 and 5.4 lead to very similar results. This is true for several reasons. The canonical form of the system and the use of identical spreads around the same nominal value make the role of  $\theta_3$  very similar to  $\theta_4$  in the transfer of the observation information to the control effort.

From these four examples we have discovered that the use of the second-order controllers can lead to large savings over the use of first-order controllers, but only in terms of the total savings possible. The losses incurred by using the first-order controllers relative to the actual coverage cost, however, is less significant. Also the two controllers become more similar as the variance increases, as in Examples 5.3 and 5.4 where they are seen to be almost identical in one case.

These observations are useful when one is considering which order to choose in the design process. Basically, if the fixed costs associated with increasing the controller dimension are greater than the resulting savings in operating costs, one should probably want to choose the first-order controller. Therefore, for higher dimensional systems, the controller used should be that which balances the fixed cost of increasing the controller order with the savings in operating costs that the increase realizes. This concludes our consideration of the design of least sensitive controllers for these examples.

### 5.3. Minimax Controllers

In this section we seek to find minimax controllers for these examples. In Chapter 4 we saw that the solution to the minimax problem, (8), is equivalent to the solution to the maximin problem, (9), if the saddle-point condition, (56), of Theorem 1 holds. We saw that this condition is equivalent to showing that

$$J_{MM}(\theta) \leq J_{MM}(\theta_o) \quad (184)$$

where  $J_{MM}(\theta)$  is the maximin cost over the parameter set and  $\theta_o$  is the maximin solution point. In Chapter 4 we saw that it is difficult to find general conditions that guarantee the satisfaction of (184). In this section, we illustrate how numerical procedures can be used to design maximin controllers and to verify (184) when the particular example is already defined.

The maximin solution occurs at the value of the unknown parameter that maximizes the optimal cost,  $J_{OPT}(\theta)$ . Therefore, the maximin solution can

be found by inspecting a plot of  $J_{OPT}(\theta)$  over the parameter range. Once the maximin solution is found, and since we can numerically check that  $\hat{S}$  in (50a) is stable in these examples, condition (184) may be proved by inspecting a plot of  $J_{MM}(\theta)$  over the parameter range.

We use this numerical method of design in Examples 5.1 through 5.4 and compare the performances of the resulting controllers to those of the least sensitive controllers.

We start with Example 5.1:

$$\theta_1 \text{ unknown, } \theta_2 = -1, \theta_3 = 1, \theta_4 = 1$$

with 10%, 30% and 50% spreads:

$$\begin{aligned} \theta_1 &\in [0.9, 1.1] \\ \theta_1 &\in [0.7, 1.3] \\ \theta_1 &\in [0.5, 1.5] \end{aligned} \quad (185)$$

A plot of  $J_{OPT}(\theta_1)$  for  $\theta_1 \in [0.2, 1.8]$  is shown in Figure 36. From this plot we see that the maxima over intervals (185) occur at

$$\begin{aligned} \theta_{10} &= 1.1 \\ \theta_{10} &= 1.3 \\ \theta_{10} &= 1.5 \end{aligned} \quad (186)$$

We can use the gradient algorithm to calculate the corresponding maximin controllers in the block diagonal coordinates. The results are:

Interval	$p_1$	$p_2$	$p_3$	$p_4$	$J_{AVG}$
[0.9,1.1]	-7.3819	-4.4681	1.4608	-6.5788	27.828
[0.7,1.3]	-8.3046	-4.7127	1.6794	-7.9796	32.853
[0.5,1.5]	-9.2709	-4.9540	1.9009	-9.6529	39.970



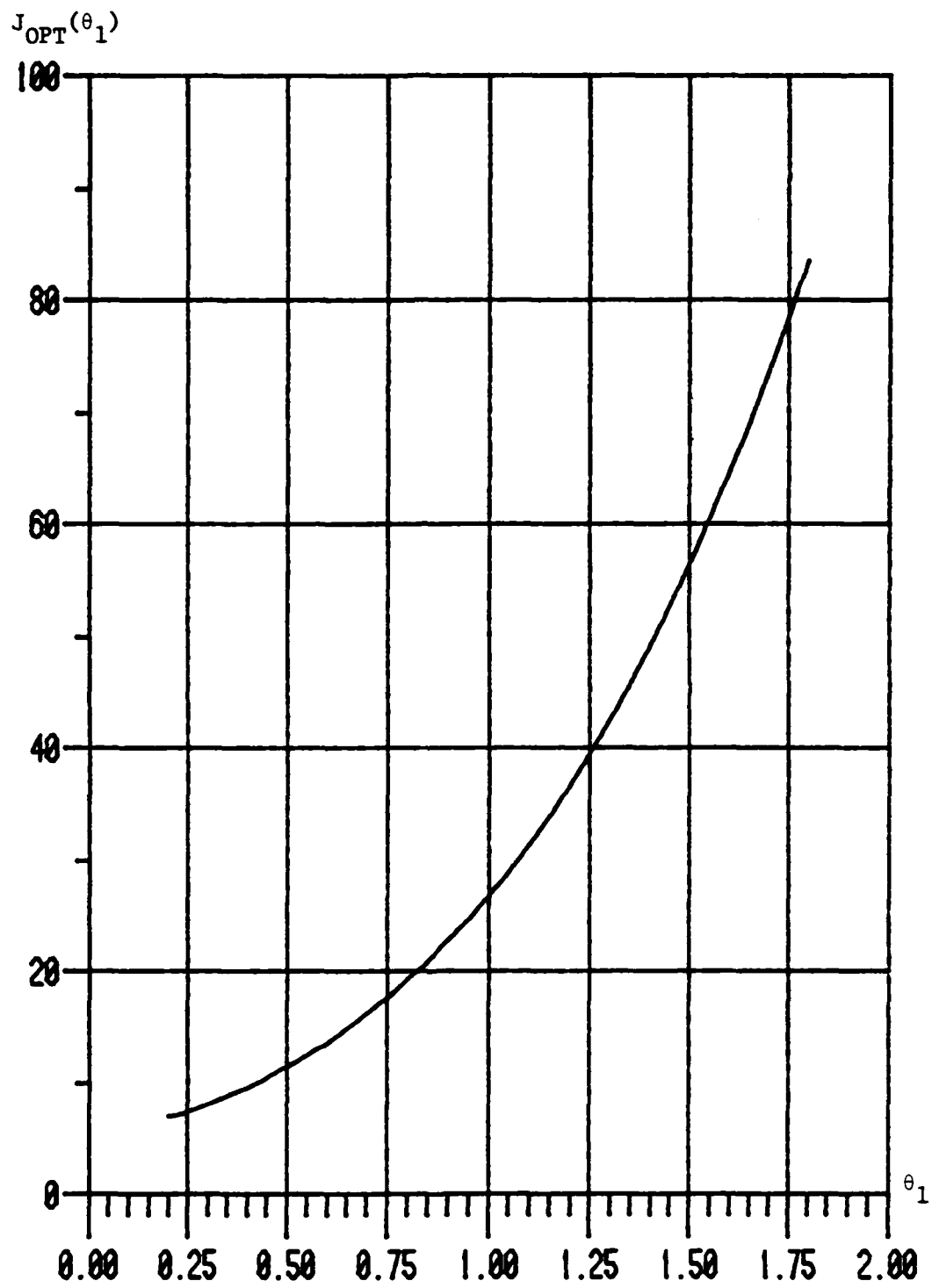


Figure 36.  $J_{OPT}$  vs.  $\theta_1$ .

A plot of  $J_{MM}(\theta_1)$  is shown in Figure 37 for all three intervals. From these curves we can see that

$$J_{MM}(\theta_1) \leq J_{MM}(\theta_{10})$$

for each of the intervals (185) and corresponding maximin points (186).

Therefore, these are minimax controllers also.

Figures 24, 25 and 26 show the plots of  $J_{MM}(\theta_1) - J_{OPT}(\theta_1)$  that we use as our typical performance measure. Note we can make similar observations as we did in the case of least sensitive controllers.

1) We calculate the decrease in average cost of the minimax controllers relative to the average cost of the first-order controllers.

Interval	Decrease in $J_{AVG}$ (Minimax)	Decrease in $J_{AVG}$ (2nd Order)
[0.9,1.1]	2.8%	5.2%
[0.7,1.3]	-4.6%	4.6%
[0.5,1.5]	-10.5%	4.2%

Note that only for the [0.9,1.1] case are the minimax controllers better than the first-order least sensitive controllers, and the minimax controllers are never better than the second-order least sensitive controllers.

2) We calculate the percentage of the maximin possible savings over the first-order controllers that the minimax controllers realize.

Interval	% of Max. Savings (Minimax)	% of Max. Savings (2nd Order)
[0.9,1.1]	45.8%	81.9%
[0.7,1.3]	-36.8%	34.6%
[0.5,1.5]	-49.5%	21.1%

Again we see that only in the [0.9,1.1] case are the minimax controllers better than the first-order controllers, and the minimax controllers are never better than the second-order controllers.

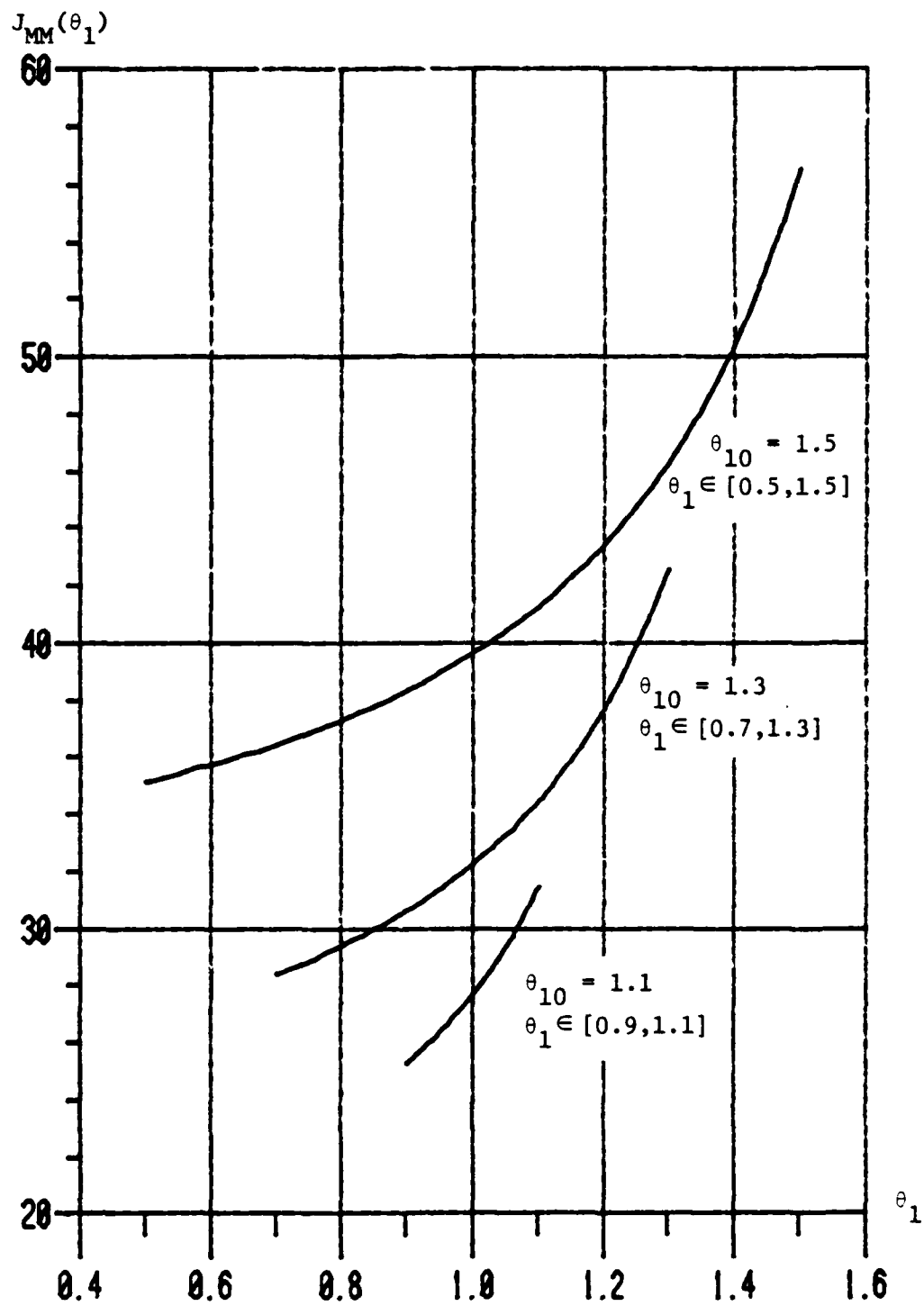


Figure 37.  $J_{MM}(\theta_1)$  vs.  $\theta_1$  for various maximin controllers.

In this example we see that the performances of the minimax controllers are again generally inferior to the performances of least sensitive controllers, even with order reduction. We see that this is also true in the other examples.

Next we consider Example 5.2:

$$\theta_1 = 1, \theta_2 \text{ unknown}, \theta_3 = 1, \theta_4 = 1$$

with 10%, 30% and 50% spreads:

$$\begin{aligned} \theta_2 &\in [-1.1, -0.9] \\ \theta_2 &\in [-1.3, -0.7] \\ \theta_2 &\in [-1.5, -0.5] \end{aligned} \quad (187)$$

A plot of  $J_{\text{OPT}}(\theta_2)$  for  $\theta_2 \in [-1.8, -0.2]$  is shown in Figure 38. From this plot we see that the maxima over intervals (187) occur at

$$\begin{aligned} \theta_{20} &= -0.9 \\ \theta_{20} &= -0.7 \\ \theta_{20} &= -0.5 \end{aligned} \quad (188)$$

We can use the gradient algorithm to calculate the corresponding maximin controllers in the block diagonal coordinates. The results are:

Interval	$p_1$	$p_2$	$p_3$	$p_4$	$J_{\text{AVG}}$
$[-1.1, -0.9]$	-7.0685	-4.3528	1.3354	-6.0950	26.927
$[-1.3, -0.7]$	-7.3982	-4.3916	1.3016	-6.8252	27.945
$[-1.5, -0.5]$	-7.8451	-4.4654	1.2704	-7.7766	29.976

A plot of  $J_{\text{MM}}(\theta_2)$  is shown in Figure 39 for all three intervals. From these curves we can see that

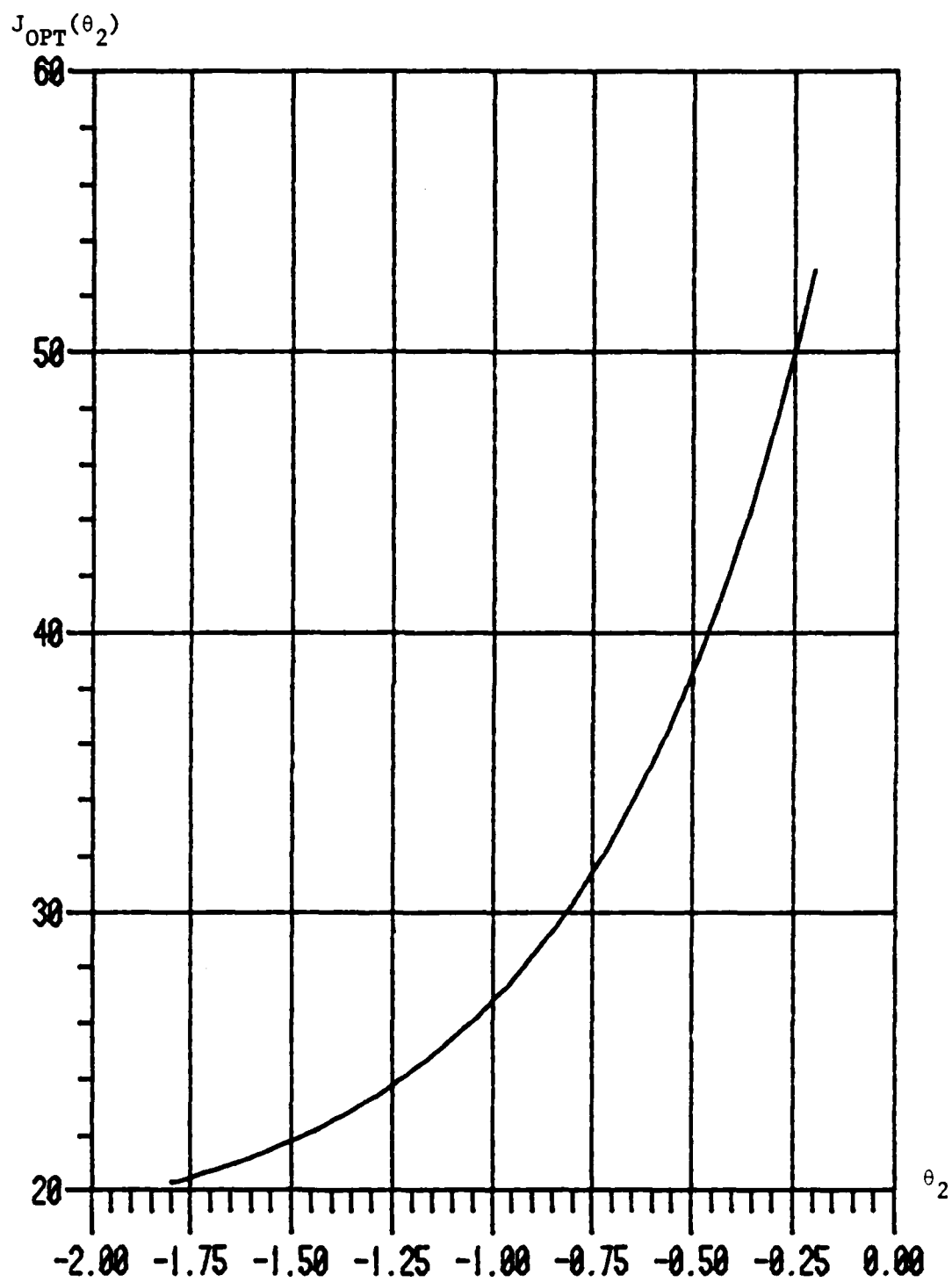


Figure 38.  $J_{OPT}$  vs.  $\theta_2$ .

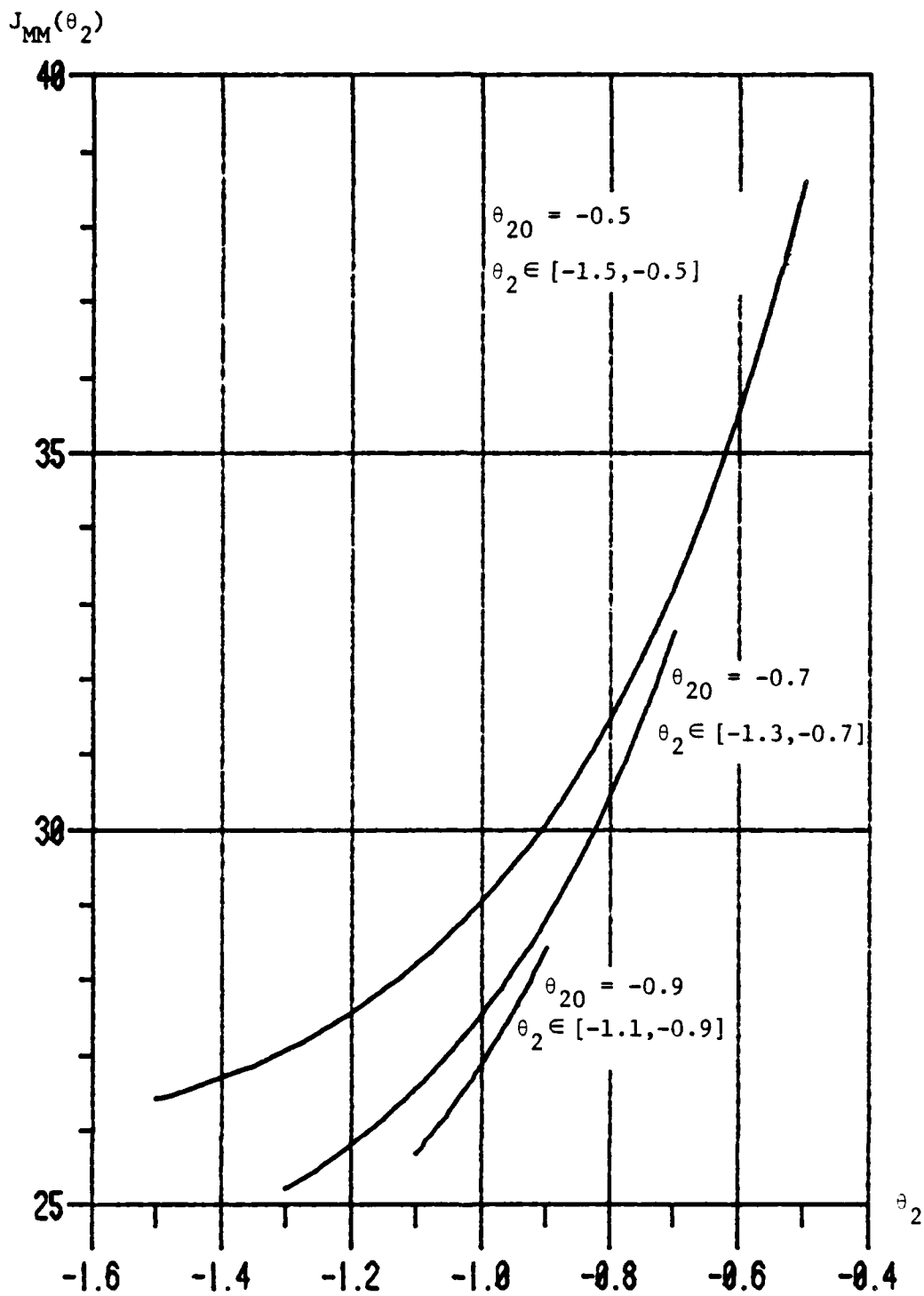


Figure 39.  $J_{MM}(\theta_2)$  vs.  $\theta_2$  for various maximin controllers.

$$J_{MM}(\theta_2) \leq J_{MM}(\theta_{20})$$

for each of the intervals (187) and corresponding maximin points (188). Therefore, these are minimax controllers also.

Figures 27, 28 and 29, show the performance of the minimax controllers. We again make the same observations.

1) We calculate the decrease in average cost of the minimax controllers relative to the average cost of the first-order controllers.

Interval	Decrease in $J_{AVG}$ (Minimax)	Decrease in $J_{AVG}$ (2nd-Order)
[-1.1,-0.9]	5.0%	5.2%
[-1.3,-0.7]	4.3%	8.1%
[-1.5,-0.5]	4.0%	8.6%

Note here that in all three cases the minimax controllers are better than the first-order least sensitive controllers, but are not as good as the second-order controllers.

2) We calculate the percentage of the maximin possible savings over the first-order controllers that the minimax controllers realize.

Interval	% of Max. Savings (Minimax)	% of Max. Savings (2nd-Order)
[-1.1,-0.9]	94.0%	98.3%
[-1.3,-0.7]	63.2%	88.7%
[-1.5,-0.5]	42.0%	80.4%

Again, we see that the minimax controller is better than the first-order controllers, but not as good as the second-order controllers.

In Examples 5.1 and 5.2 we saw that the maximin controllers were indeed equivalent to the minimax controllers. In the next two examples we see cases where this is not true.

Consider Example 5.3:

$$\theta_1 = 1, \theta_2 = -1, \theta_3 \text{ unknown}, \theta_4 = 1$$

with 10%, 30% and 50% spreads:

$$\begin{aligned} \theta_3 &\in [0.9, 1.1] \\ \theta_3 &\in [0.7, 1.3] \\ \theta_3 &\in [0.5, 1.5] \end{aligned} \quad (189)$$

A plot of  $J_{\text{OPT}}(\theta_3)$  for  $\theta_3 \in [0.2, 1.8]$  is shown in Figure 40. From this plot we see that the maxima over intervals (189) occur at

$$\begin{aligned} \theta_{30} &= 0.9 \\ \theta_{30} &= 0.7 \\ \theta_{30} &= 0.5 \end{aligned} \quad (190)$$

We can use the gradient algorithm to calculate the corresponding maximin controllers in the block diagonal coordinates. The results are:

Interval	$P_1$	$P_2$	$P_3$	$P_4$	$J_{\text{AVG}}$
[0.9, 1.1]	-6.7615	-4.2817	1.4940	-6.2355	28.420
[0.7, 1.3]	-6.4389	-4.1675	1.8976	-7.5453	41.525
[0.5, 1.5]	-6.1749	-4.0743	2.6272	-10.0216	107.885

A plot of  $J_{\text{MM}}(\theta_3)$  is shown in Figure 41 for all three intervals. From these curves we can see that

$$J_{\text{MM}}(\theta_3) \leq J_{\text{MM}}(\theta_{30})$$

only for the interval, [0.9, 1.1]. Therefore, the maximin solution corresponds to the minimax solution only for this interval. We can see from the plots of



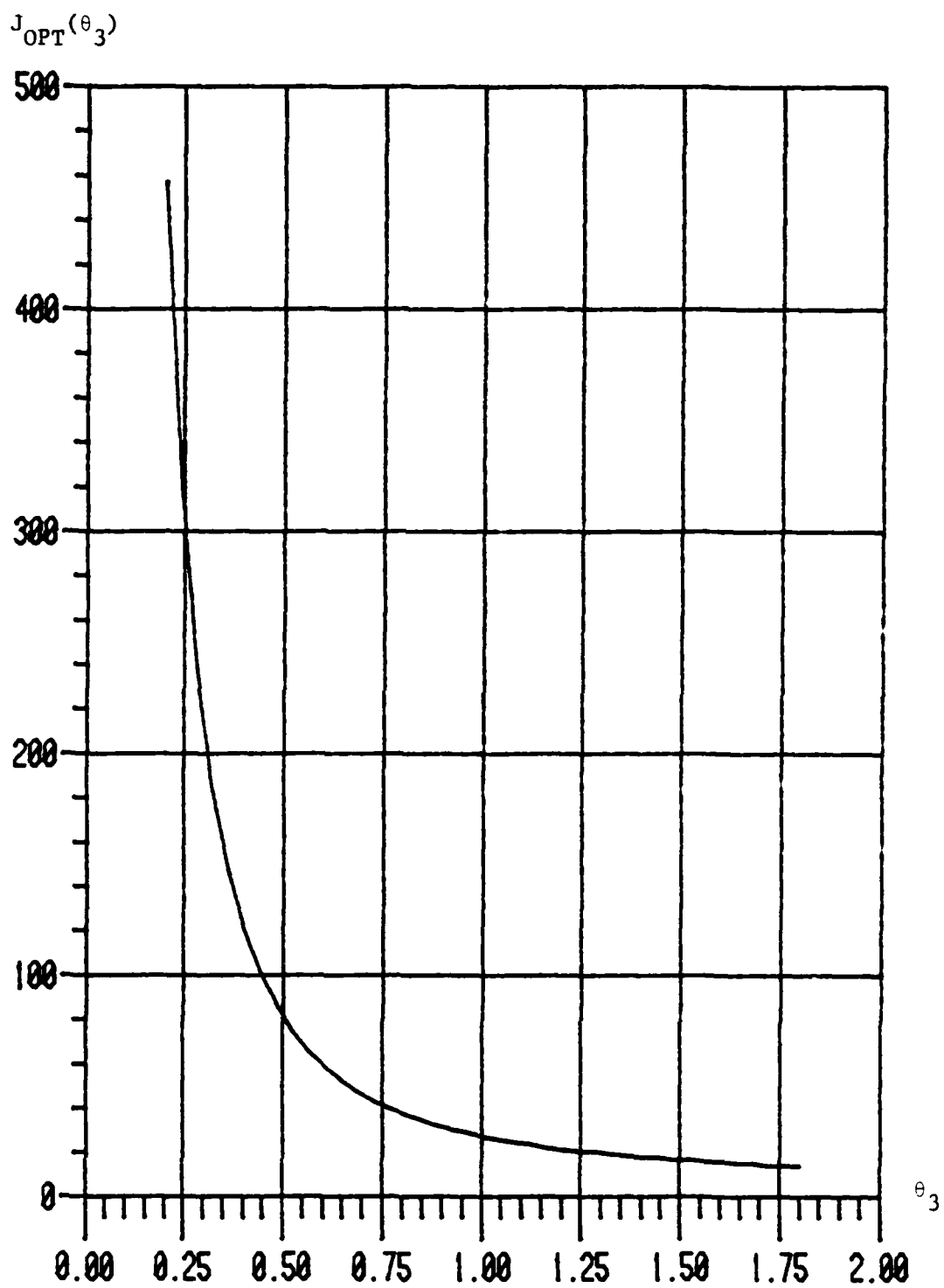


Figure 40.  $J_{OPT}$  vs.  $\theta_3$ .

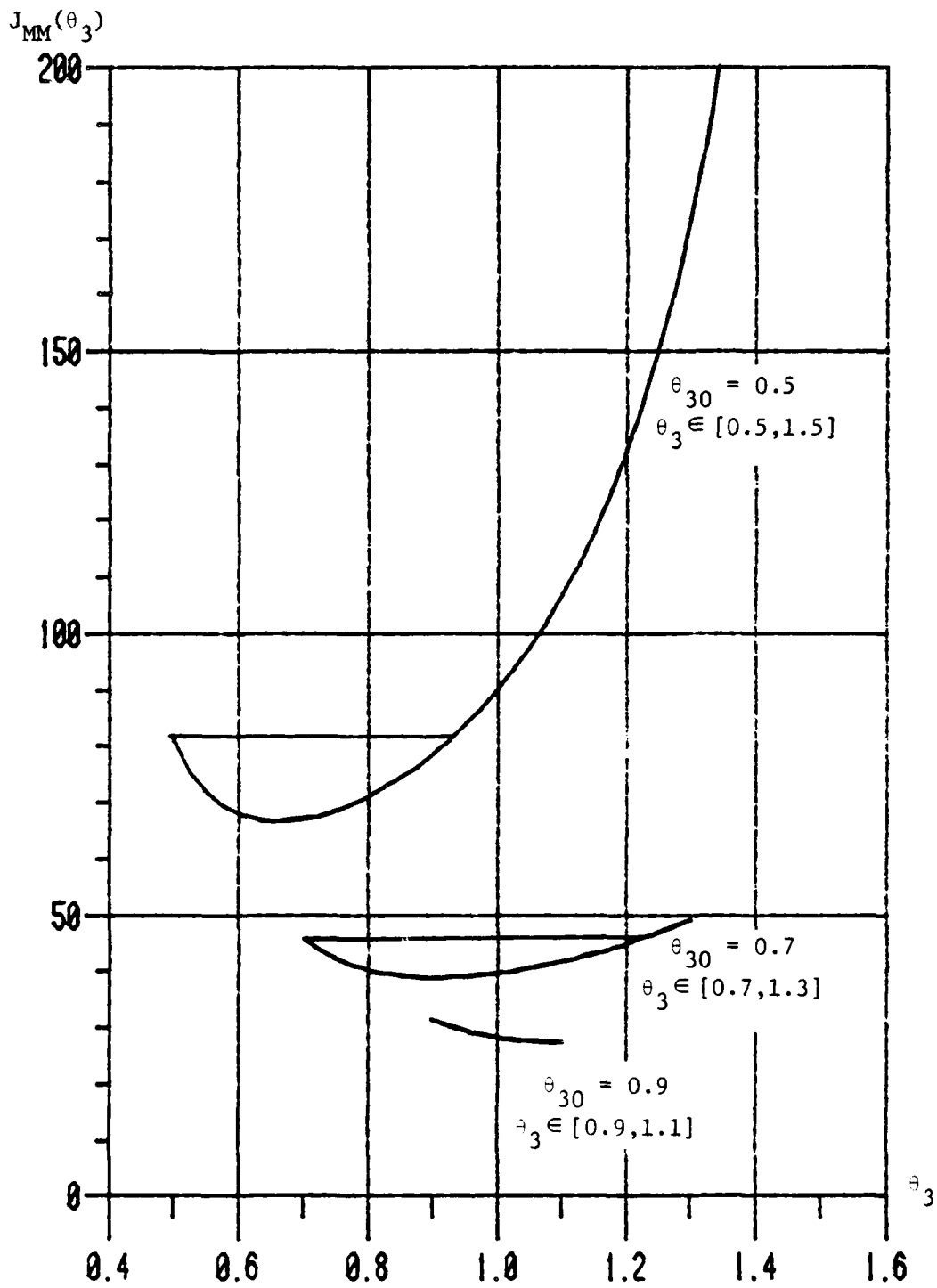


Figure 41.  $J_{MM}$  vs.  $\theta_3$  for various maximin controllers.

the performance of these controllers in Figures 30, 31 and 32 that the performance of the controllers is not good, although there are no marked dissimilarities from cases where the maximin controllers are equivalent to the minimax controllers. Again, we make our observations.

1) We calculate the decrease in average cost of the maximin controllers relative to the average cost of the first-order controllers.

Interval	Decrease in $J_{AVG}$ (Maximin)	Decrease in $J_{AVG}$ (2nd-Order)
[0.9,1.1]	1.3%	4.8%
[0.7,1.3]	-23.8%	3.3%
[0.5,1.5]	-138.2%	2.1%

As can be seen, the performance of the two maximin controllers that do not exhibit the saddle point property compare very poorly.

2) We calculate the percentage of the maximin possible savings over the first-order controllers that the maximin controllers realize.

Interval	% of Max. Savings (Maximin)	% of Max. Savings (2nd-Order)
[0.9,1.1]	19.8%	75.9%
[0.7,1.3]	-147.1%	18.1%
[0.5,1.5]	-157.5%	3.3%

Again, the poor performance of the maximin controllers that do not exhibit the saddle-point property is evident.

We should expect to find a similar situation in Example 5.4 as in Example 5.3, and indeed we do. Consider Example 5.4:

$$\theta_1 = 1, \theta_2 = -1, \theta_3 = 1, \theta_4 \text{ unknown}$$

with 10%, 30% and 50% spreads:

$$\begin{aligned}
 \theta_4 &\in [0.9, 1.1] \\
 \theta_4 &\in [0.7, 1.3] \\
 \theta_4 &\in [0.5, 1.5]
 \end{aligned}
 \tag{191}$$

A plot of  $J_{\text{OPT}}(\theta_4)$  for  $\theta_4 \in [0.2, 1.8]$  is shown in Figure 42. From this plot we see that the maxima over intervals (191) occur at

$$\begin{aligned}
 \theta_{40} &= 0.9 \\
 \theta_{40} &= 0.7 \\
 \theta_{40} &= 0.5
 \end{aligned}
 \tag{192}$$

We can use the gradient algorithm to calculate the corresponding maximin controllers in the block diagonal coordinates. The results are:

Interval	$P_1$	$P_2$	$P_3$	$P_4$	$J_{\text{AVG}}$
[0.9, 1.1]	-6.7020	-4.2678	1.4906	-6.1664	23.322
[0.7, 1.3]	-6.2585	-4.1239	1.8828	-7.2761	40.640
[0.5, 1.5]	-5.8769	-4.0005	2.5894	-9.4004	85.132

A plot of  $J_{\text{MM}}(\theta_4)$  is shown in Figure 43 for all three intervals. From these curves we can see that

$$J_{\text{MM}}(\theta_4) \leq J_{\text{MM}}(\theta_{40})$$

only for the interval, [0.9, 1.1]. Therefore, as in Example 5.3, the other two controllers are not minimax controllers, and they exhibit the same type of poor performance. Again, we make the observations.

- 1) We calculate the decrease in average cost of the maximin controllers relative to the average cost of the first-order controllers.

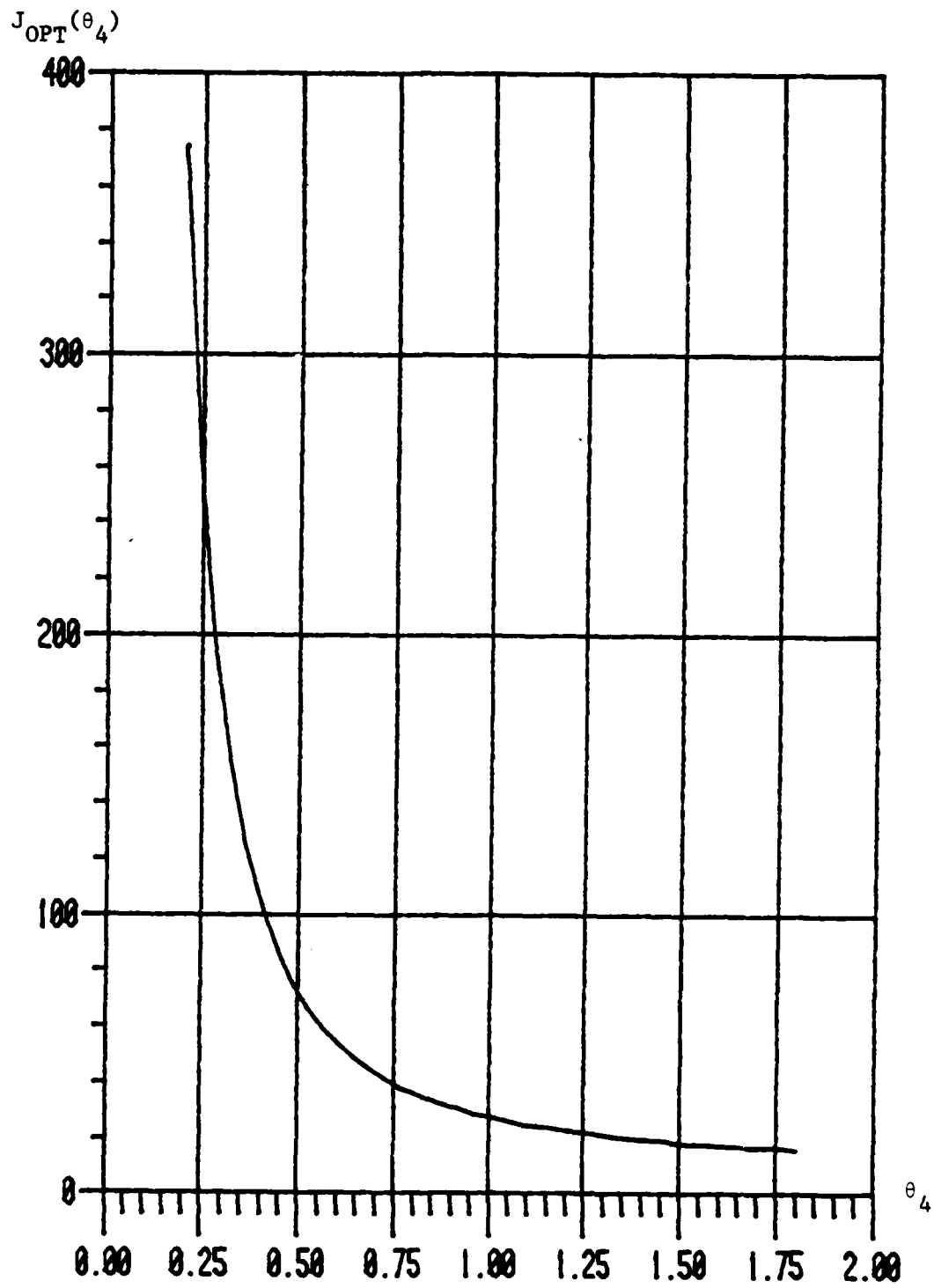


Figure 42.  $J_{OPT}$  vs.  $\theta_4$ .

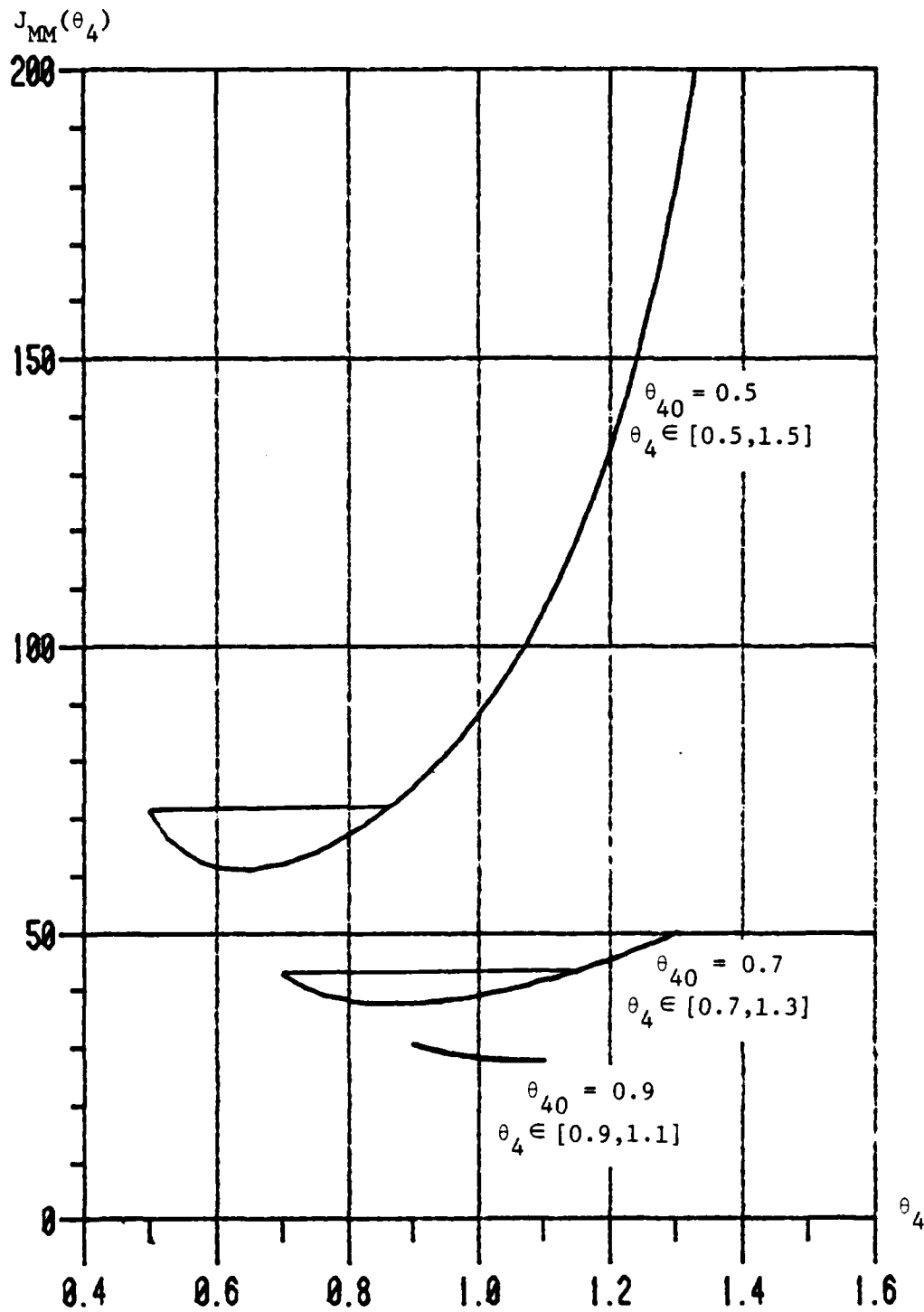


Figure 43.  $J_{MM}$  vs.  $\theta_4$  for various maximin controllers.

Interval	Decrease in $J_{AVG}$ (Maximin)	Decrease in $J_{AVG}$ (2nd-Order)
[0.9,1.1]	1.4%	4.8%
[0.7,1.3]	-23.1%	3.3%
[0.5,1.5]	-94.7%	2.1%

2) We calculate the percentage of the maximum possible savings over the first-order controllers that the maximin controllers realize.

Interval	% of Max. Savings (Maximin)	% of Max. Savings (2nd-Order)
[0.9,1.1]	22.3%	76.8%
[0.7,1.3]	-149.4%	15.2%
[0.5,1.5]	-56.9%	4.2%

As in Example 5.3, the maximin controllers that do not satisfy the saddle-point condition exhibit poor performance.

We have seen from these examples that although the minimax (or maximin) controllers are not better than the second-order least sensitive controllers, there are many cases where they are superior to the first-order least sensitive controllers, particularly when the intervals are relatively small.

Note that in few cases is the maximum cost over the interval greater for the minimax controllers than for the least sensitive controllers (see Figure 27 for an exception), so that this design consideration is not as much of an issue in these examples.

#### 5.4. Conclusions

In conclusion, the design of least sensitive and minimax controllers for a second-order system was considered in this chapter. The dimension of

the system allowed for the investigation of the effects of reduced-order least sensitive controllers, and also provided an illustration of a numerical design procedure for minimax controllers.

In the case of the design of least sensitive controllers, it was seen that some performance is sacrificed for order reduction. Furthermore, the sacrifice in performance is on the average a small percentage of the total average cost if one used an optimal controller at each value of the parameter. However, a large percentage of this additional cost can be removed using full-order controllers in these examples. The choice of controller order was seen to be a trade-off between the extra fixed costs for increasing the order and the lower operating costs this increase realizes.

In the case of the design of minimax controllers, the dimension of the system provided an illustration of a numeric design procedure that can be performed if one has a specific example. The maximin solutions were found using a plot of the optimal cost over the range of uncertainty. The saddle-point condition was verified using a plot of the cost incurred by the maximin controller over the range of uncertainty. Some examples where the saddle point did not exist were found, and the corresponding controllers were seen to exhibit poor performance. Finally, it was seen that the minimax controllers generally have poorer performance than the second-order least sensitive controllers, but in many cases have better performance than the first-order least sensitive controllers.



## CHAPTER 6

### CONCLUSIONS

This thesis has considered the design of controllers for linear stochastic systems where the corresponding models have uncertain parameters. In Chapter 1, we discussed several ways of handling these uncertainties in the design process. One method is to induce a distribution on the parameter set. Another method is to allow for variations about a nominal value. A third method is to assume that the parameter lies within a given set, and could take values which are unfavorable to the designer.

The objectives for the design of controllers for handling these uncertainties may vary. One may only desire stable performance over the whole range of parameter values. One may desire that the cost incurred for the controller at each value of the parameter be less than a certain value. Another objective may be to achieve near optimal performance for as wide a range of parameter values as possible. The two design procedures discussed in Chapters 3 and 4 are based on certain assumptions about the parameter sets, and achieve design objectives based on these assumptions.

In Chapter 3, we assumed that the parameter set is a compact set centered about a nominal value. We assumed each value equally likely; so we induced a uniform distribution on the parameter set. Our design objective was to have as near optimal performance as possible for the widest range of off nominal values of the parameter in the set. Using the assumptions on the parameter set, we were able to formulate a problem where we minimized the average cost over the entire set.

The resulting problem is a constrained optimization problem. Ashkenzai and Bryson [1] developed a method of solution when the parameter set has a finite number of values, and a discrete distribution is assumed. This procedure was extended here to solve problems when the parameter set is compact and a continuous distribution is assumed.

Some scalar examples were investigated to see what design objectives were met. It was found that:

- 1) Performance was best at values of the parameter near the nominal value, but biased towards those values that are associated with higher costs.
- 2) The performance generally worsened as the value of the parameter under consideration is further away from the nominal value.
- 3) The performance is near optimal for certain values of the parameter, but not optimal at any value, so that some performance is always sacrificed at all values of parameter in order to achieve less sensitivity.

In Chapter 4, we assumed that the unknown parameters were in convex, compact sets centered around a nominal value. Our design objective was to find the best controller for the worst-case model. Solving this problem directly is infeasible since we cannot describe all models associated with every admissible control. Another problem is to find the best control for all models, and then to find the model that exhibits the worst cost when excited by its optimal controller. The two problems, described by (8) and (9), are equivalent if a saddle point exists at the solution point of (9).

We outlined a procedure for solving the maximin problem, (9), by using the Fréchet differentials of the optimal cost function, (48). Looze, Poor, et al. [7] have shown that a saddle point exists for a general class of

uncertainty sets where the uncertainty lies in the noise covariance. We were unable to establish the same type of result when the uncertainty is in the system dynamics because the formulation of  $J(u, \theta)$  in (5) became too complicated. However, a summary of the conditions a problem must satisfy for a saddle point to exist was detailed in Theorem 2.

We then considered the equivalence of (8) and (9) for scalar systems and parameter sets that are intervals on the real line. We found cases where the saddle point does not exist, but we were able to describe all the sets satisfying the assumptions such that a saddle point does exist.

The examples of Chapter 3 were then investigated in Chapter 4 to see what design objectives were met. It was found that:

- 1) The controllers are optimal at the maximin point, so that there exist some values of the parameter such that the performance of the minimax controllers is better than the least sensitive controllers.
- 2) The performance worsens as we consider values of the parameter that are further away from the maximin point to such a degree that we found the least sensitive controllers exhibit superior performance over most of the range of uncertainty.
- 3) We found that in some cases the maximum cost for the minimax controllers is less than the maximum cost for the least sensitive controllers.

In Chapter 5, we considered a two dimensional system in order to illustrate some additional properties of these two types of controllers. First, the designs of full-order and reduced-order least sensitive controllers were considered. We saw that performance was sacrificed when the order of the controller was reduced, but this sacrifice is a small percentage of the total average cost, so that it may be desirable to use a reduced order controller

over the full order controller if the savings in the fixed costs are comparable to the additional operating costs exhibited by the reduced order controllers.

Second, a numerical procedure for designing minimax controllers for specific examples was illustrated. The maximin solution was found by investigating a plot of the optimal cost for each value of the parameter. The saddle point condition was checked using a plot of the cost incurred by the maximin controller for each value of parameter in the set. Using this procedure we found examples where the saddle point does not exist, and we saw that these maximin controllers exhibited poor performance.

We compared the performances of the minimax and maximin controllers to that of the least sensitive controllers for these examples. It was seen that in many cases the minimax controllers exhibited better performance than the first-order controllers, but were never better than the full order controllers.

The choice of minimax controllers or least sensitive controllers must be based on one's initial design objectives. If we simply desire better performance over the widest range of parameter values, the least sensitive controllers would be preferred. However, there are cases when the maximum value of the least sensitive controllers exceeds that of the minimax controllers, in which case the maximin controllers may be preferred if there is a ceiling on the maximum cost that is tolerable.

Reduced-order least sensitive controllers may be preferred over higher or full-order controllers if the saving in the fixed costs of the order reduction is comparable to the operating losses incurred by reducing the order. One avenue for future work may be to systematize the decision process of what order is most desirable for a certain class of systems.

There is also a need to obtain saddle point results for maximin controllers designed for systems of higher dimension. There may be a class of parameter sets such that a saddle point condition for each example considered may be relieved to some degree.

Furthermore, our choices of design are not necessarily restricted to the two methods considered in this thesis. One idea is to use a combination of the two designs, for instance, one can use a least sensitive controller designed for an interval that is wider than the interval under consideration, and thus can reduce the high maximum costs that are prevalent at the end points of the intervals. Of course, this sacrifices performance at other values in the interval. Therefore, one should not necessarily restrict his choices of controllers to the designs considered in this paper. Our main interest is in designing the controller that best achieves our design objectives.

## APPENDIX

## A CONJUGATE GRADIENT ALGORITHM

C This program solves the necessary conditions of  
 C the optimization problem in Chapter 3. A conjugate  
 C gradient algorithm, ZXCGR, is used from the IMSL  
 C library [6]. An explanation of the algorithm can be  
 C found in [8].

C This program first reads in the system information,  
 C then calculates an approximating discrete distribution  
 C to the uniform distribution. Then, after being given  
 C the initial values of P, the program calls the gradient  
 C routine, ZXCGR, which calls JTP for the values of JAY  
 C and its gradients, DHAM.  
 C  
 C

```

      DOUBLE PRECISION S(20,20),G(10,10),N(20,20),H(10,10),L(10,10),
A      V(20,20),W(20,20),WU(10,10),
A      PT,T(2,1000),
A      P(10),DHAM(10),WK(60),
A      ACC,MAXFN,DFPRED,JAY,AT,BT,INC,VAL,TEM,E
      INTEGER      NX,NU,NY,NZ,NW,NV,NDIST,
A      NFT,NGT,NKT,NHT,
A      FT(5,3),GT(5,3),KT(5,3),HT(5,3),
A      NP,IER,I,J
      COMMON      S(20,20),G(10,10),N(20,20),H(10,10),L(10,10),
A      V(20,20),W(20,20),WU(10,10),
A      PT,T(2,1000),
A      NX,NU,NY,NZ,NW,NV,NDIST,
A      NFT,NGT,NKT,NHT,
A      FT(5,3),GT(5,3),KT(5,3),HT(5,3)
      EXTERNAL      JTP
  
```

C This section reads in the system matrices, their  
 C dimensions, and the location of the unknown parameters  
 C in the system matrices.

C These are the dimensions of the system matrices.

```

      READ(5,*)NX,NU,NY,NZ,NW,NV,NDIST
  
```

C This reads the entries of the "F" matrix.

```

      READ(5,*)((S(I,J),J=1,NX),I=1,NX)
  
```

C This reads the location of the unknown parameters in F.

C NFT- number of parameters in F  
 C FT(I,1)- row location of parameter

```

C      FT(I,2)- column location of parameter
C      FT(I,3)- entry of T vector to be put in this location
C
C      READ(5,*)NFT,((FT(I,J),J=1,3),I=1,NFT)
C
C      Entries of the "G" matrix.
C
C      READ(5,*)((G(I,J),J=1,NU),I=1,NX)
C      READ(5,*)NGT,((GT(I,J),J=1,3),I=1,NGT)
C
C      Entries of the "K" matrix
C
C      READ(5,*)((N(I,J),J=1,NW),I=1,NX)
C      READ(5,*)NKT,((KT(I,J),J=1,3),I=1,NKT)
C
C      Entries of the "H" matrix
C
C      READ(5,*)((H(I,J),J=1,NX),I=1,NY)
C      READ(5,*)NHT,((HT(I,J),J=1,3),I=1,NHT)
C
C      The following read the entries of the L, Q, R, WX,
C      and WU matrices, respectively.
C
C      READ(5,*)((L(I,J),J=1,NV),I=1,NY)
C      READ(5,*)((V(I,J),J=1,NW),I=1,NW)
C      READ(5,*)((V(NW+I,NW+J),J=1,NV),I=1,NV)
C      READ(5,*)((W(I,J),J=1,NX),I=1,NX)
C      READ(5,*)((WU(I,J),J=1,NU),I=1,NU)
C
C      This section calculates an approximating discrete
C      distribution to the uniform distribution for a
C      one dimensional T vector.  User must supply any
C      code needed for higher dimensional T vectors.
C
C      This reads in the desired interval of T values for
C      the uniform distribution.
C
C      READ(5,*)AT,BT
C
C      This calculates the discrete distribution.
C
C      IF(NDIST.EQ.1)GOTO3
C      E=0.0
C      DO 1 I=1,NDIST-2
1      E=E+I**2
C      TEM=DSQRT(12.0*(1.0/NDIST-.25+E/(NDIST*(NDIST-1)**2)))
C      INC=(BT-AT)/TEM/(NDIST-1)
C      VAL=(AT+BT)/2-(BT-AT)/2/TEM
C      DO 2 I=1,NDIST
C      T(1,I)=VAL
2      VAL=VAL+INC
C      GOTO4
3      T(1,1)=AT

```

C  
C This section provides the initial value of P for  
C the local gradient algorithm, ZXCGR, and also some  
C other initial values the routine needs. See Reference  
C [6] for details on the usage of ZXCGR.

C  
4 PT=1.0/NDIST  
DO 5 I=1,NX  
DO 5 J=1,NX  
5 W(I,J)=PT\*W(I,J)  
NP=(NU+NY)\*NZ  
READ(5,\*)(P(I),I=1,NP)  
READ(5,\*)DFPRED  
ACC=.1e-9  
MAXFN=20  
CALL ZXCGR(JTP,NP,ACC,MAXFN,DFPRED,P,DHAM,JAY,WK,IER)

C  
C ZXCGR yields the stationary values of P, the value of the  
C cost, JAY, and the gradients, DHAM, at those values of  
C the vector, P.

C  
PRINT100,(P(I),I=1,NP)  
PRINT100,(DHAM(I),I=1,NP)  
100 FORMAT(2X,5(E14.8,2X)/)  
PRINT\*,JAY  
STOP  
END

C  
C  
C SUBROUTINE JTP(NP,P,JAY,DHAM)

C  
C This subroutine calculates JAY, and its gradients, DHAM,  
C which is the information needed for the IMSL conjugate  
C gradient algorithm, ZXCGR [6].

C  
DOUBLE PRECISION S(20,20),G(10,10),N(20,20),H(10,10),L(10,10),  
A V(20,20),W(20,20),WU(10,10),  
A PT,T(2,1000),  
A B(10,10),C(10,10),DB(10,10),DC(10,10),  
A DS(20,20),DN(20,20),DW(20,20),  
A COV(20,20),LAM(20,20),  
A ALPHT(190,190),RESB(190,190),RVCR(190),RVLR(190),  
A RES(20,20),RESA(20,20),RVC(190),RVL(190),  
A P(NP),DHAM(NP),JAY,work(200)  
INTEGER NX,NU,NY,NZ,NW,NV,NDIST,  
A NFT,NGT,NKT,NHT,  
A FT(5,3),GT(5,3),KT(5,3),HT(5,3),  
A NP,NZT,NS,NN,NVEC,  
A I,J,IA,IB,IC,ID,IP,M,LM,KK,II,LL,JJ,IER  
COMMON S(20,20),G(10,10),N(20,20),H(10,10),L(10,10),  
A V(20,20),W(20,20),WU(10,10),  
A PT,T(2,1000),



```

A      NX,NU,NY,NZ,NW,NV,NDIST,
A      NFT,NGT,NKT,NHT,
A      FT(5,3),GT(5,3),KT(5,3),HT(5,3)

```

```

C
C      Some initializations that are needed.
C

```

```

      NS=NX+NZ
      NN=NW+NV
      NVEC=NS*(NS+1)/2
      NZT=NZ/2*2
      JAY=0
      DO 6 I=1,NP
      DHAM(I)=0

```

```

6
C
C      This section puts the values of P(IP) into the
C      appropriate entries of A(P), B(P), and C(P).
C

```

```

      IP=0
      DO 1 I=2,NZ,2
      IP=IP+1
      S(NX+I,NX+I-1)=P(IP)
      IP=IP+1
      S(NX+I,NX+I)=P(IP)
1      S(NX+I-1,NX+I)=1
      IF(NZT.EQ.NZ)GOTO2
      IP=IP+1
      S(NS,NS)=P(IP)
2      DO 3 I=1,NZ
      DO 3 J=1,NY
      IP=IP+1
3      B(I,J)=P(IP)
      DO 4 I=2,NZ,2
4      C(1,I)=1
      IF(NZT.NE.NZ)C(1,NZ)=1
      DO 5 I=2,NU
      DO 5 J=1,NZ
      IP=IP+1
5      C(I,J)=P(IP)
C

```

```

C      This loop sums JAY and DHAM for all the values of T
C      in the distribution that was passed from the main
C      program.
C

```

```

      DO 1000 ID=1,NDIST

```

```

C
C      This section puts the values of T into the appropriate
C      entries of F, G, K, and H, according to the positions
C      indicated by FT, GT, KT, HT, respectively.
C

```

```

      DO 10 I=1,NFT
10      S(FT(I,1),FT(I,2))=T(FT(I,3),ID)
      DO 11 I=1,NGT
11      G(GT(I,1),GT(I,2))=T(GT(I,3),ID)

```

```

      DO 12 I=1,NKT
12     N(KT(I,1),KT(I,2))=T(KT(I,3),ID)
      DO 13 I=1,NHT
13     H(HT(I,1),HT(I,2))=T(HT(I,3),ID)
C
C     This section calculates S, W, and NVN'.
C     These are needed to solve the Lyapunov equations,
C     (24), and (27).
C
      CALL VMULFF(G,C,NX,NU,NZ,10,10,RES,20,IER)
      DO 20 I=1,NX
      DO 20 J=1,NZ
20     S(I,NX+J)=RES(I,J)
      CALL VMULFF(B,H,NZ,NY,NX,10,10,RES,20,IER)
      DO 21 I=1,NZ
      DO 21 J=1,NX
21     S(NX+I,J)=RES(I,J)
      CALL VMULFF(B,L,NZ,NY,NV,10,10,RES,20,IER)
      DO 22 I=1,NZ
      DO 22 J=1,NV
22     N(NX+I,NW+J)=RES(I,J)
      CALL VMULFM(C,WU,NU,NZ,NU,10,10,RESA,20,IER)
      CALL VMULFF(RESA,C,NZ,NU,NZ,20,10,RES,20,IER)
      DO 23 I=1,NZ
      DO 23 J=1,NZ
23     W(NX+I,NX+J)=PT*RES(I,J)
      CALL VMULFF(N,V,NS,NN,NN,20,20,RESA,20,IER)
      CALL VMULFP(RESA,N,NS,NN,NS,20,20,RES,20,IER)
C
C     This section rearranges the entries of S into a matrix,
C     ALPHT, so that the equations, (24) and (27), are of the
C     form, "Ax=B", which is suitable for inversion. See
C     Reference [4].
C
      M=0
      DO 70 II=1,NS
      DO 70 JJ=II,NS
      M=M+1
      IF(II.EQ.JJ)GOTO78
      RVC(M)=-RES(II,JJ)
      RVL(M)=-W(II,JJ)
      GOTO79
78     RVC(M)=-RES(II,JJ)/2
      RVL(M)=-W(II,JJ)/2
79     LM=0
      DO 70 KK=1,NS
      DO 70 LL=KK,NS
      LM=LM+1
      IF(KK.EQ.II.AND.LL.NE.JJ)GOTO71
      IF(KK.NE.II.AND.LL.EQ.JJ)GOTO72
      IF(KK.NE.II.AND.LL.NE.JJ.AND.KK.EQ.JJ.AND.LL.NE.II)GOTO73
      IF(KK.NE.II.AND.LL.NE.JJ.AND.KK.NE.JJ.AND.LL.EQ.II)GOTO74
      IF(KK.NE.II.AND.LL.NE.JJ.AND.KK.NE.JJ.AND.LL.NE.II)GOTO75

```

```

      IF(KK.EQ.II.AND.LL.EQ.JJ.AND.KK.EQ.JJ.AND.LL.EQ.II)GOTO76
      IF(KK.EQ.II.AND.LL.EQ.JJ.AND.KK.NE.JJ.AND.LL.NE.II)GOTO77
71    ALPHT(M,LM)=S(LL,JJ)
      GOTO70
72    ALPHT(M,LM)=S(KK,II)
      GOTO70
73    ALPHT(M,LM)=S(LL,II)
      GOTO70
74    ALPHT(M,LM)=S(KK,JJ)
      GOTO70
75    ALPHT(M,LM)=0
      GOTO70
76    ALPHT(M,LM)=S(KK,II)
      GOTO70
77    ALPHT(M,LM)=S(KK,II)+S(LL,JJ)
70    CONTINUE

```

```

C
C    This subroutine inverts ALPHT so that COV and LAM
C    can be calculated. A warning - LINV2F often will
C    not work for smaller sizes of ALPHT (dimension 3X3).
C

```

```

      CALL LINV2F(ALPHT,NVEC,190,RESB,12,WORK,IER)

```

```

C
C    This section calculates COV and LAM using the inverted
C    ALPHT.
C

```

```

      CALL VMULFF(RESB,RVL,NVEC,NVEC,1,190,190,RVLR,190,IER)
      CALL VMULFM(RESB,RVC,NVEC,NVEC,1,190,190,RVCR,190,IER)
      IP=0
      DO 31 I=1,NS
      DO 31 J=I,NS
      IP=IP+1
      COV(I,J)=RVCR(IP)
      COV(J,I)=RVCR(IP)
      LAM(I,J)=RVLR(IP)
31    LAM(J,I)=RVLR(IP)

```

```

C
C    This section calculates JAY=tr[W*COV].
C

```

```

      CALL VMULFF(W,COV,NS,NS,NS,20,20,RES,20,IER)
      DO 32 I=1,NS
32    JAY=JAY+RES(I,I)

```

```

C
C    This section calculates the gradients of JAY, DHAM,
C    for the P's that are entries of the "A(P)" matrix.
C    For the "A" parameters-
C

```

```

      DHAM=tr{[LAM*DS+DS'*LAM]*COV}

```

```

C
      IP=0
      DO 40 I=2,NZ,2
      IP=IP+1
      DS(NX+I,NX+I-1)=1

```

```

CALL VMULFF(DS, COV, NS, NS, NS, 20, 20, RESA, 20, IER)
CALL VMULFF(RESA, LAM, NS, NS, NS, 20, 20, RES, 20, IER)
DO 41 IA=1, NS
41 DHAM(IP)=DHAM(IP)+2*RES(IA, IA)
   DS(NX+I, NX+I-1)=0
   IP=IP+1
   DS(NX+I, NX+I)=1
   CALL VMULFF(DS, COV, NS, NS, NS, 20, 20, RESA, 20, IER)
   CALL VMULFF(RESA, LAM, NS, NS, NS, 20, 20, RES, 20, IER)
   DO 42 IA=1, NS
42 DHAM(IP)=DHAM(IP)+2*RES(IA, IA)
40 DS(NX+I, NX+I)=0
   IF(NZ T.EQ.NZ)GOTO44
   IP=IP+1
   DS(NS, NS)=1
   CALL VMULFF(DS, COV, NS, NS, NS, 20, 20, RESA, 20, IER)
   CALL VMULFF(RESA, LAM, NS, NS, NS, 20, 20, RES, 20, IER)
   DO 43 IA=1, NS
43 DHAM(IP)=DHAM(IP)+2*RES(IA, IA)
   DS(NS, NS)=0
C
C   This section calculates the gradients of JAY, DHAM,
C   for the P's that are entries of the "B(P)" matrix.
C   For the "B" parameters -
C
C   
$$DHAM = tr\{[LAM*DS + DS' * LAM] * COV\}$$

C   
$$+ tr\{LAM * [DN * V * N' + N * V * DN']\}$$

C
44 DO 50 I=1, NZ
   DO 50 IA=1, NY
   IP=IP+1
   DB(I, IA)=1
   CALL VMULFF(DB, L, NZ, NY, NV, 10, 10, RES, 20, IER)
   DO 53 IB=1, NZ
   DO 53 IC=1, NV
53 DN(NX+IB, NW+IC)=RES(IB, IC)
   CALL VMULFF(DN, V, NS, NN, NN, 20, 20, RESA, 20, IER)
   CALL VMULFF(RESA, N, NS, NN, NS, 20, 20, RES, 20, IER)
   CALL VMULFF(N, V, NS, NN, NN, 20, 20, RESB, 190, IER)
   CALL VMULFF(RESB, DN, NS, NN, NS, 190, 20, RESA, 20, IER)
   DO 54 IB=1, NS
   DO 54 IC=1, NS
54 RES(IB, IC)=RES(IB, IC)+RESA(IB, IC)
   CALL VMULFF(LAM, RES, NS, NS, NS, 20, 20, RESA, 20, IER)
   DO 55 IB=1, NS
55 DHAM(IP)=DHAM(IP)+RESA(IB, IB)
   CALL VMULFF(DB, H, NZ, NY, NX, 10, 10, RES, 20, IER)
   DO 56 IB=1, NZ
   DO 56 IC=1, NX
56 DS(NX+IB, IC)=RES(IB, IC)
   CALL VMULFF(DS, COV, NS, NS, NS, 20, 20, RESA, 20, IER)
   CALL VMULFF(RESA, LAM, NS, NS, NS, 20, 20, RES, 20, IER)
   DO 57 IB=1, NS

```

```

57   DHAM(IP)=DHAM(IP)+2*RES(IB,IB)
      DO 58 IB=1,NZ
      DO 59 IC=1,NV
59   DN(NX+IB,NW+IC)=0
      DO 58 IC=1,NX
58   DS(NX+IB,IC)=0
50   DB(I,IA)=0
C
C   This section calculates the gradients of JAY, DHAM,
C   for the P's that are entries of the "C(P)" matrix.
C   For the "C" parameters -
C
C       DHAM=tr{[DW+LAM*DS+DS'*LAM]*COV}
C
      DO 60 I=2,NU
      DO 60 J=1,NZ
      IP=IP+1
      DC(I,J)=1
      CALL VMULFM(DC,WU,NU,NZ,NU,10,10,RESA,20,IER)
      CALL VMULFF(RESA,C,NZ,NU,NZ,20,10,RES,20,IER)
      DO 61 IA=1,NZ
      DO 61 IB=1,NZ
61   DW(NX+IA,NX+IB)=RES(IA,IB)
      CALL VMULFM(C,WU,NU,NZ,NU,10,10,RESA,20,IER)
      CALL VMULFF(RESA,DC,NZ,NU,NZ,20,10,RES,20,IER)
      DO 67 IA=1,NZ
      DO 67 IB=1,NZ
67   DW(NX+IA,NX+IB)=DW(NX+IA,NX+IB)+RES(IA,IB)
      CALL VMULFF(DW,COV,NS,NS,NS,20,20,RES,20,IER)
      DO 62 IA=1,NS
62   DHAM(IP)=DHAM(IP)+PT*RES(IA,IA)
      DO 63 IA=1,NZ
      DO 63 IB=1,NZ
63   DW(NX+IA,NX+IB)=0
      CALL VMULFF(G,DC,NX,NU,NZ,10,10,RES,20,IER)
      DO 64 IA=1,NX
      DO 64 IB=1,NZ
64   DS(IA,NX+IB)=RES(IA,IB)
      CALL VMULFF(DS,COV,NS,NS,NS,20,20,RESA,20,IER)
      CALL VMULFF(RESA,LAM,NS,NS,NS,20,20,RES,20,IER)
      DO 65 IA=1,NS
65   DHAM(IP)=DHAM(IP)+2*RES(IA,IA)
      DO 66 IA=1,NX
      DO 66 IB=1,NZ
66   DS(IA,NX+IB)=0
60   DC(I,J)=0
C
C   Continue with another value of T from the distribution
C   provided by the main program.
1000 CONTINUE
      RETURN
      END

```

## REFERENCES

- [1] A. Ashkenzai and A. E. Bryson, Jr., "Control Logic for Parameter Insensitivity and Disturbance Attenuation," Journal of Guidance, Control, and Dynamics, Vol. 5, No. 4, pp. 383-388, July-August 1982.
- [2] A. V. Balakrishnan, Applied Functional Analysis, pp. 54-55, New York: Springer-Verlag, 1976.
- [3] T. Başar, Stochastic Control of Linear Systems in the Continuous Time, Supplementary Notes for EE 455, Department of Electrical Engineering, University of Illinois at Urbana-Champaign, 1982.
- [4] C. F. Chen and L. S. Shieh, "A Note on Expanding  $PA + A^TP = -Q$ ," IEEE Transactions on Automatic Control, Vol. AC-13, No. 1, pp. 122-123, February 1968.
- [5] R. F. Curtain and A. J. Pritchard, Functional Analysis in Modern Applied Mathematics, pp. 28-30, New York: Academic Press, Inc., 1977.
- [6] IMSL Subroutine ZXCGR, IMSL Library Reference Manual, Houston: IMSL, Inc., 1982.
- [7] D. P. Looze, H. V. Poor, K. S. Vastola and J. C. Darragh, "Minimax Control of Linear Stochastic Systems with Noise Uncertainty," IEEE Transactions on Automatic Control, Vol. AC-28, No. 9, pp. 882-887, September 1983.
- [8] M. J. D. Powell, "Restart Procedures for the Conjugate Gradient Method," Mathematical Programming, Vol. 12, pp. 241-254, 1977.

**END**

**FILMED**

---

**1-86**

**DTIC**